

## A remark on the Omori-Yau maximum principle

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### ABSTRACT

A Riemannian manifold  $M$  is said to satisfy the Omori-Yau maximum principle if for any  $C^2$  bounded function  $g : M \rightarrow \mathbb{R}$  there is a sequence  $x_n \in M$ , such that  $\lim_{n \rightarrow \infty} g(x_n) = \sup_M g$ ,  $\lim_{n \rightarrow \infty} |\nabla g(x_n)| = 0$  and  $\limsup_{n \rightarrow \infty} \Delta g(x_n) \leq 0$ . It is shown that if the Ricci curvature does not approach  $-\infty$  too fast the manifold satisfies the Omori-Yau maximum principle. This improves earlier necessary conditions. The given condition is quite optimal.

**Keywords:** Maximum principle; Ricci curvature.

### INTRODUCTION

**Definition** A Riemannian manifold  $M$  is said to satisfy the Omori-Yau maximum principle if for any  $C^2$  function  $g : M \rightarrow \mathbb{R}$  which is bounded from above and for any  $\epsilon > 0$  there is a point  $x_\epsilon \in M$ , such that  $|g(x_\epsilon) - \sup_M g| < \epsilon$ ,  $|\nabla g(x_\epsilon)| < \epsilon$  and  $\Delta g(x_\epsilon) < \epsilon$ .

This principle has turned out to be very useful in differential geometry and received considerable attention recently. A necessary condition in terms of the Ricci curvature for a manifold to satisfy this principle was first proved by Omori (1967) and later generalized by Yau (1975). It states that if the Ricci curvature is bounded from below then the manifold satisfies the Omori-Yau maximum principle. This was improved upon in Theorem 2.3 of Ratto, *et al.* (1995).

**Theorem (Ratto-Rigoli-Setti).** Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that away from the cut locus of  $p$  we have

$$\text{Ricc}(\nabla r, \nabla r) \geq -(n-1)BG^2(r),$$

where  $B > 0$  is some constant and  $G(t)$  has the following properties:

- (i)  $G(0) = 1, \quad G' \geq 0$
- (ii)  $\int_0^\infty \frac{dt}{G(t)} = \infty$
- (iii)  $\frac{d^{2k+1}}{dt^{2k+1}} \sqrt{G}(0) = 0 \quad \text{for all } k \in \mathbb{N}$
- (iv)  $\limsup_{t \rightarrow \infty} \frac{t \sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} < \infty.$

Then  $M^n$  satisfies the Omori-Yau maximum principle.

The goal of the present note is to improve the necessary condition given in Ratto, *et al.* (1995). The actual statement is given as a Corollary. Basically we remove the last two conditions on the function  $G(t)$ , which turned out not to be essential.

Another interesting necessary condition, requiring the existence of an exhaustion function with certain properties, was given by Kim & Lee (2007). Interestingly there is an alternative proof by Kim and Lee of the Ratto-Rigoli-Setti result in Kim & Lee (2007) which is still using these extra conditions.

The proof uses the same method we used in an earlier paper Borbely (2008).

**Theorem.** Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that

$$\Delta r(x) \leq G(r(x))$$

for all  $x \in M^n$  where  $r$  is smooth and  $r(x) > 1$ , where  $G(t)$  has the following properties:

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} = \infty.$$

Then  $M^n$  satisfies the Omori-Yau maximum principle.

As a consequence we have the following.

**Corollary.** Let  $M^n$  be a complete Riemannian manifold,  $p \in M^n$  be a fixed point and  $r(x)$  be the distance function from  $p$ . Let us assume that away from the cut locus of  $p$  we have

$$\text{Ricc}(\nabla r, \nabla r) \geq -G^2(r),$$

where  $G(t)$  has the following properties:

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} = \infty.$$

Then  $M^n$  satisfies the Omori-Yau maximum principle.

The main condition on the function  $G(t)$  in the Corollary and in the Ratto-Rigoli-Setti Theorem is the same ( $\int 1/G(t) = \infty$ ) but there are additional technical conditions imposed on the function  $G(t)$  in the later Theorem. In this respect, the Corollary can be considered as a refinement of the Ratto-Rigoli-Setti Theorem.

Let us mention that this condition is quite optimal. If  $\int_0^\infty 1/G(t)dt < \infty$ , there are manifolds with  $\Delta r \leq G(r)$  for which the Omori-Yau maximum principle does not apply. The details can be found in Section 3.

### Proof of the Theorem

Set  $L = \sup g$  and let us assume that  $g < L$  at every point of  $M$ . Otherwise  $g$  assumes its maximum at some point and that point trivially satisfies the conditions of the Definition for all  $\epsilon > 0$ .

Define the function  $F(t)$  as

$$F(t) = e^{\int_0^t \frac{1}{G(s)} ds}.$$

Then clearly:  $F \geq 1$ ,  $F$  is strictly increasing and  $\lim_{t \rightarrow \infty} F(t) = \infty$ .

For any  $\epsilon < \min\{1, L - \sup\{g(x) : r(x) < 1\}\}$  define the function  $h_\lambda : M \rightarrow R$  as

$$h_\lambda(x) = \lambda F(r(x)) + L - \epsilon.$$

Since  $F(r(x)) \geq 1$ , for  $\lambda > \epsilon$  we have

$$h_\lambda(x) > L > g(x) \quad \text{for all } x \in M.$$

Define  $\lambda_0$  as

$$\lambda_0 = \inf\{\lambda : h_\lambda(x) > g(x) \quad \text{for all } x \in M\}.$$

Since  $\sup g = L$  it is easy to see that  $\lambda_0 > 0$  and  $h_{\lambda_0}(x) \geq g(x)$  for all  $x \in M$ .

We claim that there is a point  $x_\epsilon \in M$  such that  $h_{\lambda_0}(x_\epsilon) = g(x_\epsilon)$ .

This will follow from the observation that if  $h_\lambda(x) > g(x)$  for all  $x \in M$ , then there is a  $\lambda' < \lambda$  such that  $h_{\lambda'}(x) > g(x)$  for all  $x \in M$ . To show this we argue as follows.

Let  $r_0$  be large enough such that  $h_\lambda(x) > L + 1$  for  $r(x) > r_0$ . Since  $\lim_{r \rightarrow \infty} F(r) = \infty$  such  $r_0$  must exist. The set  $\{x \in M : r(x) \leq r_0\}$  is compact, therefore  $h_\lambda(x) > g(x)$  for all  $x \in M$  implies that there is a  $\lambda' < \lambda$  such that  $h_{\lambda'}(x) > g(x)$  for all  $x \in M : r(x) \leq r_0$ . Choosing  $\lambda'$  sufficiently close to  $\lambda$  we can achieve that  $h_{\lambda'}(x) > L$  for  $r(x) = r_0$ . Since  $F$  is increasing we obtain that  $h_{\lambda'}(x) > L$  for  $r(x) \geq r_0$ . Combining this with the previous remark we have  $h_{\lambda'}(x) > g(x)$  for all  $x \in M$ .

Next, we have to show that  $h_{\lambda_0}$  is smooth at  $x_\epsilon$ . The argument is exactly the same as the argument in Borbely (2008), but we include it at the end of this proof for the convenience of the reader.

Once the smoothness of  $h_{\lambda_0}$  at  $x_\epsilon$  is established, the rest of the argument is straight forward.

From the definition of  $F$  and from the fact that  $G' \geq 0$  we have

$$F' = \frac{F}{G} \quad \text{and} \quad F'' = \frac{F'}{G} - \frac{FG'}{G^2} \leq \frac{F}{G^2}.$$

From the fact that  $g(x_\epsilon) = \lambda_0 F(r(x_\epsilon)) + L - \epsilon < L$  we conclude that

$$L - g(x_\epsilon) \leq \epsilon, \tag{1.1}$$

moreover

$$\lambda_0 F(r(x_\epsilon)) < \epsilon \quad \text{hence} \quad \lambda_0 < \frac{\epsilon}{F(r(x_\epsilon))} < \epsilon. \tag{1.2}$$

Since

$$h_{\lambda_0}(x) \geq g(x), \quad \text{and} \quad h_{\lambda_0}(x_\epsilon) = g(x_\epsilon),$$

we have

$$\nabla g(x_\epsilon) = \nabla h_{\lambda_0}(x_\epsilon) \quad \text{and} \quad \Delta h_{\lambda_0}(x_\epsilon) \geq \Delta g(x_\epsilon).$$

Taking into consideration (1.2), the definition of  $F$ , the fact that  $|\nabla r| = 1$  and the assumption that  $G(r) \geq 1$ , the first equality above yields

$$|\nabla g(x_\epsilon)| = |\lambda_0 F(r(x_\epsilon)) \nabla r(x_\epsilon)| = \frac{\epsilon}{F(r)} \cdot \frac{F(r)}{G(r)} < \epsilon. \tag{1.3}$$

For the Laplace of  $h_{\lambda_0}$  we have

$$\begin{aligned} \Delta g(x_\epsilon) &\leq \Delta h_{\lambda_0}(x_\epsilon) = \lambda_0 \left( F'(r(x_\epsilon)) \Delta r(x_\epsilon) + F''(r(x_\epsilon)) |\nabla r(x_\epsilon)|^2 \right) \leq \\ &\leq \frac{\epsilon}{F} \left( \frac{F}{G} \Delta r + \frac{F}{G^2} \right) \leq 2\epsilon. \end{aligned} \tag{1.4}$$

The inequalities (1.1), (1.3) and (1.4) show that the point  $x_\epsilon$  satisfies the conditions in the Definition.

Finally, we have to show that  $h_{\lambda_0}$  is smooth at  $x_\epsilon$ . Since  $h_\lambda(x) = \lambda F(r(x)) + L - \epsilon$  it is enough to show that  $r$  is smooth at  $x_\epsilon$ . If not, then  $x_\epsilon$  must be on the cut locus of  $p$ . In this case we have two possibilities. Either there are two distinct minimizing geodesic segments  $\gamma_1, \gamma_2 : [0, t_0] \rightarrow M$  joining  $p$  to  $x_\epsilon$ , or there is a geodesic segment  $\gamma : [0, t_0] \rightarrow M$  from  $p$  to  $x_\epsilon$  along which  $x_\epsilon$  is conjugate to  $p$ .

In both cases we have

$$t_0 = r(x_\epsilon).$$

Let us start with the first case. Let  $w = \gamma_1'(t_0)$  and  $v = \gamma_2'(t_0)$ . Since  $\gamma_1$  and  $\gamma_2$  are distinct segments we have  $w \neq v$ . The functions  $t \rightarrow r(\gamma_i(t))$  are differentiable on  $(0, t_0)$  (for  $i = 1, 2$ ) and they have a left-derivative at  $t_0$ .

From the fact that  $h_{\lambda_0} \geq g$  and  $h_{\lambda_0}(x_\epsilon) = g(x_\epsilon)$  we have

$$\liminf_{s \rightarrow 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq D_v g(x_\epsilon),$$

where  $D_v g(x_\epsilon)$  denotes the directional derivative of  $g$  at the point  $x_\epsilon$  in the direction of  $v$ . Moreover since  $g$  is smooth and  $h_{\lambda_0}$  has a directional derivative at  $x_\epsilon$  in the direction of  $-v$ , we also have

$$-\lambda_0 F'(r(x_\epsilon)) = D_{-v} h_{\lambda_0}(x_\epsilon) \geq D_{-v} g(x_\epsilon) = -D_v g(x_\epsilon).$$

This yields

$$D_v g(x_\epsilon) \geq \lambda_0 F'(r(x_\epsilon)). \tag{1.5}$$

Combining this with the above inequality we obtain

$$\liminf_{s \rightarrow 0^+} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq \lambda_0 F'(r(x_\epsilon)).$$

Taking into account the special form of  $h_{\lambda_0}$  we have

$$\liminf_{s \rightarrow 0^+} \frac{r(\gamma_2(t_0 + s)) - r(\gamma_2(t_0))}{s} \geq 1. \quad (1.6)$$

This will lead to a contradiction. Since  $v \neq w$ , there is a  $0 < c < 1$  depending only on the angle of  $v$  and  $w$ , such that

$$r(\gamma_2(t_0 + s)) < t_0 + cs, \quad (1.7)$$

for a small enough  $s > 0$ .

One can see this by connecting the point  $\gamma_1(t_0 - s)$  to  $\gamma_2(t_0 + s)$  by a geodesic segment. Since  $\gamma_1$  and  $\gamma_2$  are different there is a  $0 < c_1 < 1$  such that for a small enough  $s > 0$  we have  $\text{dist}(\gamma_1(t_0 - s), \gamma_2(t_0 + s)) < c_1 2s$  and this implies (1.7). Since  $r(x_\epsilon) = r(\gamma_2(t_0)) = t_0$  it is easy to see that (1.6) and (1.7) are in direct contradiction.

We now turn our attention to the second case. Since  $\gamma$  is distance minimizing between  $p$  and  $x_\epsilon$  the distance function  $r$  is smooth at  $\gamma(t)$  for  $0 < t < t_0$ . Set  $m(t) = \Delta r(\gamma(t))$ . Then  $m(t)$  is also smooth on the interval  $(0, t_0)$  and since  $\gamma(t_0)$  is conjugate to  $p = \gamma(0)$  along  $\gamma$  we have

$$\lim_{t \rightarrow t_0^-} m(t) = -\infty. \quad (1.8)$$

Since  $\lambda_0 > 0$ , from (1.5) we conclude that  $D_v g(x_\epsilon) > 0$ , that is  $\nabla g(x_\epsilon) \neq 0$ . This implies that the level surface  $H = \{x \in M : g(x) = g(x_\epsilon)\}$  is a smooth hyper-surface near  $x_\epsilon$ . Denote by  $H_s$  the surface parallel to  $H$  and passing through the point  $\gamma(t_0 - s)$  for some  $s > 0$ . Again, since  $H$  is smooth near  $x_\epsilon$  the surface  $H_s$  will also be smooth near  $\gamma(t_0 - s)$  for a small enough  $s > 0$ .

It is now clear from (1.8) that for some small  $s > 0$  we have

$$m(t_0 - s) < \text{trace of the 2nd fundamental form of } H_s \text{ at } \gamma(t_0 - s),$$

where the second fundamental form of  $H_s$  at  $\gamma(t_0 - s)$  is taken in the direction of  $\gamma'(t_0 - s)$ .

Taking into account that  $m(t_0 - s)$  is the trace of the 2nd fundamental form of the geodesic ball  $B_p(t_0 - s)$  around  $p$  at the point  $\gamma(t_0 - s)$  (with respect to the same normal vector  $\gamma'(t_0 - s)$ ) we conclude that there has to be a point  $q_s \in H_s$ , sufficiently close to  $\gamma(t_0 - s)$ , that lies inside  $B_p(t_0 - s)$ . This means that

$$r(q_s) < t_0 - s.$$

Since  $H_s$  is parallel to  $H$  we have a point on  $q \in F$  such that  $\text{dist}(q_s, q) = s$ . Combining this with the above inequality we have

$$r(q) < t_0 = r(x_\epsilon).$$

Taking into account that  $F$  is strictly increasing we obtain

$$h_{\lambda_0}(q) = \lambda_0 F(r(q)) + L - \epsilon < \lambda_0 F(r(x_\epsilon)) + L - \epsilon = h_{\lambda_0}(x_\epsilon) = g(x_\epsilon) = g(q).$$

This leads to a contradiction since  $h_{\lambda_0} \geq g$  on  $M$ .

### Proof of the Corollary

Let  $q \in M$  be a point away from the cut locus of  $p$  and  $\gamma$  be a geodesic segment parameterized by arc length connecting  $p$  to  $q$ . Set  $m(t) = \Delta r(\gamma(t))$  and  $R(t) = \text{Ricc}(\gamma'(t), \gamma'(t))$ . Then it is well known that  $m(t)$  satisfies the Riccati inequality along  $\gamma$ . Taking into consideration the condition on the Ricci curvature we have

$$m'(t) \leq -R(t) - \frac{m^2(t)}{n-1} \leq G^2(t) - \frac{m^2(t)}{n-1}.$$

This implies that  $m$  is decreasing as long as  $m > \sqrt{n-1}G$  and a simple argument shows that

$$m(t) < (\sqrt{n-1} + 1)G,$$

for all  $t > t_0$ , where  $t_0$  is a sufficiently large constant, independent of  $G$ .

This yields

$$\Delta r < (\sqrt{n-1} + 1)G \quad \text{if } r > t_0,$$

for points that are not on the cut locus of  $p$ . Since  $(\sqrt{n-1} + 1)G$  satisfies the conditions in the Theorem the proof of the corollary is complete.

### An example

In this section we sketch an example, that shows that the condition in the Theorem is quite optimal. Let  $M^n$  be a Hadamard manifold that is rotationally symmetric around  $p \in M^n$ .

Let  $r$  be the distance function from  $p$  and assume that  $\Delta r(x) > G(r)$  for all  $x \in M^n$ , where  $G$  satisfies the conditions:

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} < \infty.$$

Then there is a bounded function  $h : M \rightarrow R$  which shows that the manifold  $M^n$  does not satisfy the Omori-Yau maximum principle. To construct  $h$  we need the following lemma.

**Lemma .** Let  $G : [0, \infty) \rightarrow R$  be a function satisfying the conditions:

$$G \geq 1, \quad G' \geq 0, \quad \text{and} \quad \int_0^\infty \frac{dt}{G(t)} < \infty.$$

Then there is a function  $H : [0, \infty) \rightarrow R$  such that

$$H \geq 1/2, \quad H' \geq 0, \quad 2H \leq G, \quad H' \leq H^2 \quad \text{and} \quad \int_0^\infty \frac{dt}{H(t)} < \infty.$$

First we construct the function  $h : M^n \rightarrow R$  and give the proof of the Lemma later.

Let

$$h(x) = \int_0^{r(x)} \frac{dt}{H(t)}.$$

The last condition on  $H$  in the Lemma implies that  $h$  is bounded from above. A simple computation shows that

$$\Delta h = \frac{\Delta r}{H} - \frac{H'}{H^2} |\nabla r|^2.$$

Since  $\Delta r > G(r) \geq 2H(r)$ ,  $|\nabla r| = 1$  and  $H' \leq H$  we have

$$\Delta h > 2 - 1 = 1.$$

This clearly shows that the manifold  $M^n$  does not satisfy the Omori-Yau maximum principle.

All that remains is to prove the Lemma.

*Proof of Lemma.* Let  $A \subset (0, \infty)$  be defined as

$$A = \left\{ t > 0 : \frac{G'(t)}{2} > \left( \frac{G(t)}{2} \right)^2 \right\}.$$



It is an open set therefore

$$A = \cup I_n,$$

where  $I_n = (t_n, s_n)$  are disjoint open intervals.

This is the set where  $G/2$  grows too fast. We obtain  $H$  by modifying  $G/2$  on a slightly larger set so that it will never grow too fast, that is  $H' \leq H^2$ .

For a given  $n$  define the function  $k_n(t)$  to be

$$k_n(t) = \frac{1}{a_n - t},$$

where  $a_n$  is chosen such that  $k_n(t_n) = G(t_n)/2$ . Then we have

$$k_n(t_n) = \frac{G(t_n)}{2}, \quad k'_n(t) = k_n^2(t) \quad \text{and} \quad \frac{G'(t)}{2} > \left(\frac{G(t)}{2}\right)^2 \quad \text{for} \quad t \in (t_n, \min\{s_n, a_n\}).$$

This implies that

$$k_n(t) < \frac{G(t)}{2} \quad \text{for} \quad t \in (t_n, \min\{s_n, a_n\}).$$

Let  $v_n > t_n$  be the first point where  $k_n(v_n) = G(v_n)/2$ . Such point must exist since  $\lim_{t \rightarrow a_n} k_n(t) = \infty$ . Therefore we have  $t_n < s_n < v_n < a_n$  and as a result  $J_n = (t_n, v_n) \supset I_n$ .

The intervals  $I_n$  are all disjoint but  $J_n$  are not necessarily disjoint intervals. However if  $J_n \cap J_m \neq \emptyset$ , then either  $J_n \subset J_m$  or  $J_m \subset J_n$ . This follows simply from the way the intervals  $J_n$  were constructed and from the fact that the graphs of the functions  $1/(a-t)$ ,  $t < a$  and  $1/(b-t)$ ,  $t < b$  are translates of each other.

Therefore we can select a pairwise disjoint family of intervals  $J_{n_i}$  such that  $B = \cup J_n = \cup J_{n_i}$ . To simplify the notation without loss of generality we can assume that the intervals  $J_n$  are already pairwise disjoint.

We can now define the function  $H(t)$  as follows

$$H(t) = \begin{cases} \frac{G(t)}{2} & \text{if } t \notin B = \cup J_n \\ \frac{1}{a_n - t} & \text{if } t \in J_n. \end{cases}$$

It is clear from the construction that  $H$  satisfies the first four properties in the Lemma. It remains to be shown that it will satisfy the remaining property

$$\int_0^{\infty} \frac{dt}{H(t)} < \infty. \quad (3.1)$$

We can write

$$\int_0^{\infty} \frac{dt}{H(t)} = \int_B \frac{dt}{H(t)} + \int_{R^+ - B} \frac{dt}{H(t)}.$$

The second integral is clearly finite since

$$\int_{R^+ - B} \frac{dt}{H(t)} = \int_{R^+ - B} \frac{2dt}{G(t)} < \infty.$$

The first integral can be computed as follows

$$\int_B \frac{dt}{H(t)} = \int_{\cup J_n} \frac{dt}{H(t)} = \sum_{n=1}^{\infty} \int_{t_n}^{v_n} a_n - t \, dt = \frac{1}{2} \sum_{n=1}^{\infty} (a_n - t_n)^2 - (a_n - v_n)^2.$$

From the construction of the intervals  $J_n$  and the function  $H$  one obtains that

$$a_n - t_n = \frac{2}{G(t_n)} \quad \text{and} \quad a_n - v_n = \frac{2}{G(v_n)}.$$

To show that the infinite sum above is finite it is enough to show that any partial-sum is bounded by a fixed constant. For this reason consider the sum

$$\sum_{n=1}^m (a_n - t_n)^2 - (a_n - v_n)^2 = \sum_{n=1}^m \left( \frac{2}{G(t_n)} \right)^2 - \left( \frac{2}{G(v_n)} \right)^2.$$

By rearranging the terms if necessary, without loss of generality we can assume that

$$t_1 < v_1 < t_2 < v_2 < \dots < t_n < v_n < t_{n+1} < v_{n+1} < \dots < t_m < v_m.$$

Taking into consideration that  $G(t)$  is an increasing function, we obtain that

$$\sum_{n=1}^m (a_n - t_n)^2 - (a_n - v_n)^2 = \sum_{n=1}^m \left( \frac{2}{G(t_n)} \right)^2 - \left( \frac{2}{G(v_n)} \right)^2 < \left( \frac{2}{G(t_1)} \right)^2.$$

This shows that the above sum is finite, which in turn proves (3.1). This completes the proof of the lemma.

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## ملاحظة حول مبدأ الحد الأقصى أوموري - يو

البرت بوريلي

قسم الرياضيات - كلية العلوم - جامعة الكويت - الكويتا

### خلاصة

نقول إن منظوياً ريمانياً  $M$  يحقق مبدأ الحد الأقصى أوموري - يو إذا كان من أجل أي دالة محدودة  $g: M \rightarrow \mathbb{R}$  من الصنف  $C_2$ ، يوجد متتالية  $x_n \in M$ ، بحيث:

$$\liminf(x_n) = \sup g, \quad \lim \text{gradient} g(x_n) = 0 \quad \text{and} \quad \limsup(x_n) \leq 0.$$

ونبين هنا أنه إذا لم يقترب تقوس ريتشي من  $1$  أكثر مما ينبغي، فإن المنظوي يحقق مبدأ الحد الأقصى أوموري - يو.

وهذا يحسن الشروط اللازمة التي سبق الحصول عليها. أما الشرط المفروض فهو أمثلي تماماً.