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Identities leading to joint distributions of order statistics of *innid* variables from a truncated distribution

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ABSTRACT

In this study, the joint distributions of order statistics from *innid* variables from a truncated distribution is expressed in terms of the specialized identities. Then, some results connecting distributions of order statistics from *innid* variables from a truncated distribution to that of order statistics from *iid* variables from a truncated distribution are given.

Keywords: Order statistics; truncated distribution; permanent; joint distribution; *innid* random variable.

MSC 2010: 62G30, 62E15

INTRODUCTION

Several identities and recurrence relations for probability density function (*pdf*) and distribution function (*df*) of order statistics of independent and identically distributed (*iid*) random variables were established by numerous authors including Arnold *et al.* (1992); Balasubramanian & Beg (2003); David (1981); and Reiss (1989). Furthermore, Arnold *et al.* (1992), David (1981); Gan & Bain (1995), and Khatri (1962) obtained the probability function and *df* of order statistics of *iid* random variables from a discrete parent. Corley (1984) defined a multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie & Maller (1997). Guilbaud (1982) expressed the probability of the functions of independent but not necessarily identically distributed (*innid*) random vectors as a linear combination of probabilities of the functions of *iid* random vectors and thus also for order statistics of random variables.

Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao & West (1997). In addition, Vaughan and Venables (1972) derived the joint *pdf* and marginal *pdf* of order statistics of *innid* random variables by means of permanents. Balakrishnan (2007); and Bapat & Beg (1989) obtained the joint *pdf* and *df* of order statistics of *innid* random variables by means of permanents. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ was obtained by Childs & Balakrishnan (2006) by adding another independent random variable to the original n variables $X_1, X_2, ..., X_n$. Also, Balasubramanian *et al.* (1994) established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. Beg (1991) obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer *et al.* (2009) derived the expressions for the distribution and density functions by Ryser's method and the distributions of maxima and minima based on permanents. In the first of two papers, Balasubramanian *et al.* (1991) obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian *et al.* (1996) generalized their previous results (1991) to the case of the joint distribution function of several order statistics. Güngör *et al.* (2009) examined distributions of order statistics. The joint distributions of order statistics of *innid* random variables are expressed by Güngör (2012). In this study, the joint distributions of order statistics from *innid* variables from a truncated distribution are obtained. From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

If $\mathrm{a}_1, \mathrm{a}_2, ...$are defined as column vectors, then the matrix obtained by taking m_1 copies of a_1, m_2 copies of $\mathrm{a}_2, ...$ can be denoted as

$$\underset{m_1}{[\mathrm{a}_1}\ \underset{m_2}{\mathrm{a}_2} \ldots]$$

and *per* A denotes the permanent of a square matrix A, which is defined as similar to determinant except that all terms in the expansion have a positive sign.

Let F_i and f_i $(i = 1, 2, ..., n)$ be *df* and *pdf* of X_i, respectively.

$$\alpha(F_i) = \inf\{x : F_i(x) > 0\} = F_i^{-1}(0) \text{ and } \omega(F_i) = \sup\{x : F_i(x) < 1\} = F_i^{-1}(1), \quad x \in R. \quad (1)$$

From (1), we can write

$$\alpha({}_{uv}F_i) = u_i \quad \text{and} \quad \omega({}_{uv}F_i) = v_i \quad (2)$$

From (2), *df* and *pdf* of X_i which truncated on the left at u_i and right at v_i, respectively, are expressed as

$$_{uv}F_i(x) = \frac{F_i(x) - F_i(u_i)}{F_i(v_i) - F_i(u_i)} \quad \text{and} \quad _{uv}f_i(x) = \frac{f_i(x)}{F_i(v_i) - F_i(u_i)} \ . \tag{3}$$

Let $X_1, X_2, ..., X_n$ be *innid* continuous random variables which are truncated and $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Moreover, $X^s_{1:n}$, $X^s_{2:n}$, ... , $X^s_{n:n}$ are order statistics of truncated random variables with common *df* $_{uv}F^s$ and *pdf* $_{uv}f^s$, respectively, defined by

$$_{uv}F^s = \frac{1}{n_s}\sum_{i\in s} {_{uv}F_i} \tag{4}$$

and

$$_{uv}f^s = \frac{1}{n_s}\sum_{i\in s} {_{uv}f_i}. \tag{5}$$

Here, s is a subset of the integers $\{1, 2,... , n\}$ with $n_s \geq 1$ elements. A[s/.) is the matrix obtained from A by taking rows whose indices are in s.

The *df* and *pdf* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_d:n}$ $(1 \leq r_1 < r_2 < ... < r_d \leq n, d = 1, 2, ..., n)$

will then be given. For notational convenience we write $\sum\sum$, $\sum\limits_{m_d,...,m_2,m_1}^{n,...,m_3,m_2}$, $\sum\limits_{t_d,...,t_2,t_1}^{n,...,m_3,m_2}$ and $\sum\limits_{t_d,...,t_2,t_1}^{n,...,r_3-1,r_2-1}$ instead of $\sum\limits_{\kappa=1}^{n}(-1)^{n-\kappa}\frac{\kappa^n}{n!}\sum\limits_{n_s=\kappa}$, $\sum\limits_{m_d=r_d}^{n} ... \sum\limits_{m_2=r_2}^{m_3}\sum\limits_{m_1=r_1}^{m_2}$, $\sum\limits_{t_d=m_d}^{n} ... \sum\limits_{t_2=m_2}^{m_3}\sum\limits_{t_1=m_1}^{m_2}$ and $\sum\limits_{t_d=r_d}^{n} ... \sum\limits_{t_2=r_2}^{r_3-1}\sum\limits_{t_1=r_1}^{r_2-1}$ in the expressions below, respectively.

2. IDENTITIES LEADING TO DISTRIBUTION FUNCTION

The identities in the following theorems will be used to obtain joint *df* of order statistics from *innid* variables from a truncated distribution. The identities connect order statistics of *innid* random variables to that of *iid* random variables using (4). We will now express five theorems using to establish the *df* of order statistics from *innid* variables from a truncated distribution.

Theorem 2.1.

$$per[\underset{m_1}{\Delta_{\text{uv}}\text{F}(x_1)} \quad \underset{m_2-m_1}{\Delta_{\text{uv}}\text{F}(x_2)} \quad ... \quad \underset{n-m_d}{\Delta_{\text{uv}}\text{F}(x_{d+1})}] = \sum_P \prod_{w=1}^{d+1} \prod_{l=m_{w-1}+1}^{m_w} \Delta_{\text{uv}}F_{j_l}(x_w), x_1 < x_2 < ... < x_d, \tag{6}$$

where $\Delta_{uv}\mathrm{F}(x_w) = (\Delta_{uv}F_1(x_w),\ \Delta_{uv}F_2(x_w), \ldots,\ \Delta_{uv}F_n(x_w))'$ ($w = 1,2,\ldots,d+1$) is column vector, $x_w \in R$, $\sum_P$ denotes the sum over all $n!$ permutations $(j_1, j_2, \ldots, j_n)$ of $(1,2,\ldots,n)$, $m_0 = 0$, $m_{d+1} = n$, $\Delta_{uv}F_{j_l}(x_w) = {}_{uv}F_{j_l}(x_w) - {}_{uv}F_{j_l}(x_{w-1})$, ${}_{uv}F_{j_l}(x_0) = 0$ and ${}_{uv}F_{j_l}(x_{d+1}) = 1$.

Proof. Using expansion of permanent, it can be written:

$$per[\underset{m_1}{\Delta_{uv}\mathrm{F}(x_1)} \quad \underset{m_2-m_1}{\Delta_{uv}\mathrm{F}(x_2)} \quad \ldots \quad \underset{n-m_d}{\Delta_{uv}\mathrm{F}(x_{d+1})}]$$
$$= \sum_P \Delta_{uv}F_{j_1}(x_1)\ldots\Delta_{uv}F_{j_{m_1}}(x_1)\ \Delta_{uv}F_{j_{m_1+1}}(x_2)\ldots\Delta_{uv}F_{j_{m_2}}(x_2)\ldots\Delta_{uv}F_{j_{m_d+1}}(x_{d+1})\ldots\Delta_{uv}F_{j_n}(x_{d+1})$$
$$= \sum_P \left(\prod_{l=1}^{m_1} \Delta_{uv}F_{j_l}(x_1)\right)\left(\prod_{l=m_1+1}^{m_2} \Delta_{uv}F_{j_l}(x_2)\right)\ldots \prod_{l=m_d+1}^{n} \Delta_{uv}F_{j_l}(x_{d+1}).$$

Thus, (6) is obtained.

The following theorem connects the *df* of order statistics of *innid* random variables to that of order statistics of *iid* random variables.

Theorem 2.2.

$$\sum_P \prod_{w=1}^{d+1} \prod_{l=m_{w-1}+1}^{m_w} \Delta_{uv}F_{j_l}(x_w) = \sum\sum n! \prod_{w=1}^{d+1} [\Delta_{uv}F^s(x_w)]^{m_w - m_{w-1}}, \qquad (7)$$

where ${}_{uv}F^s(x_{d+1}) = 1$.

Proof. The proof was omitted because of a special case of the proof of Theorem 2.1.

Identity in Theorem 2.2 can be expressed as (8) using binomial expansion.

Theorem 2.3.

$$\begin{aligned} &\sum\sum n! \prod_{w=1}^{d+1} [\Delta_{uv}F^s(x_w)]^{m_w - m_{w-1}} \\ &= \sum\sum n! \left[{}_{uv}F^s(x_1)\right]^{m_1} \prod_{w=2}^{d+1} \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w - t} \binom{m_w - m_{w-1}}{t - m_{w-1}} \left[{}_{uv}F^s(x_w)\right]^{t - m_{w-1}} \left[{}_{uv}F^s(x_{w-1})\right]^{m_w - t}. \end{aligned} \qquad (8)$$

Proof. From binomial expansion, (8) is obtained.

Theorem 2.4.

$$\sum\sum n! \left[{}_{uv}F^s(x_1)\right]^{m_1} \prod_{w=2}^{d+1} \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w - t} \binom{m_w - m_{w-1}}{t - m_{w-1}} \left[{}_{uv}F^s(x_w)\right]^{t - m_{w-1}} \left[{}_{uv}F^s(x_{w-1})\right]^{m_w - t}$$

$$= \sum\sum n! \sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1} - t_w)} \prod_{w=1}^{d} \binom{m_{w+1} - m_w}{t_w - m_w} \left[{}_{uv}F^s(x_w)\right]^{n_{s_w}}, \qquad (9)$$

where $n_{s_w} = m_{w+1} - m_{w-1} - t_w + t_{w-1}$ and $t_0 = m_1$.

Proof.

$$\sum\sum n!\,[{}_{\text{uv}}F^s(x_1)]^{m_1}\prod_{w=2}^{d+1}\sum_{t=m_{w-1}}^{m_w}(-1)^{m_w-t}\binom{m_w-m_{w-1}}{t-m_{w-1}}[{}_{\text{uv}}F^s(x_w)]^{t-m_{w-1}}[{}_{\text{uv}}F^s(x_{w-1})]^{m_w-t}$$

$$=\sum\sum n![{}_{uv}F^s(x_1)]^{m_1}\sum_{t_1=m_1}^{m_2}(-1)^{m_2-t_1}\binom{m_2-m_1}{t_1-m_1}[{}_{uv}F^s(x_2)]^{t_1-m_1}[{}_{uv}F^s(x_1)]^{m_2-t_1}\sum_{t_2=m_2}^{m_3}(-1)^{m_3-t_2}\binom{m_3-m_2}{t_2-m_2}$$

$$.[{}_{uv}F^s(x_3)]^{t_2-m_2}[{}_{uv}F^s(x_2)]^{m_3-t_2}\ \dots\ \sum_{t_d=m_d}^{n}(-1)^{n-t_d}\binom{n-m_d}{t_d-m_d}[{}_{uv}F^s(x_d)]^{n-t_d}$$

$$=\sum\sum n!\sum_{t_d=m_d}^{n}\dots\sum_{t_2=m_2}^{m_3}\sum_{t_1=m_1}^{m_2}(-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)}\prod_{w=1}^{d}\binom{m_{w+1}-m_w-}{t_w-m_w}[{}_{uv}F^s(x_w)]^{n_{s_w}}.$$

Thus, (9) is obtained.

Identity (1) can be expressed as (10) using properties of permanent.

Theorem 2.5.

$$per[\underset{m_1}{{}_{\text{uv}}\mathrm{F}(x_1)}\quad \underset{m_2-m_1}{{}_{\text{uv}}\mathrm{F}(x_2)-{}_{\text{uv}}\mathrm{F}(x_1)}\quad \dots\quad \underset{n-m_d}{1-{}_{\text{uv}}\mathrm{F}(x_d)}]$$

$$=\sum_{t_d,\dots,t_2,t_1}^{n,\dots,m_3,m_2}(-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)}\left[\prod_{w=1}^{d}\binom{m_{w+1}-m_w}{t_w-m_w}\right]\sum_{n_s=n-t_d+m_d}(t_d-m_d)!$$

$$.per[\underset{m_2-t_1+m_1}{{}_{\text{uv}}\mathrm{F}(x_1)}\quad \underset{m_3-m_1-t_2+t_1}{{}_{\text{uv}}\mathrm{F}(x_2)}\quad \dots\quad \underset{n-m_{d-1}-t_d+t_{d-1}}{{}_{\text{uv}}\mathrm{F}(x_d)}]\quad [s/.)\ , \qquad (10)$$

where ${}_{uv}\mathrm{F}(x_w) - {}_{uv}\mathrm{F}(x_{w-1}) = ({}_{\text{uv}}F_1(x_w) - {}_{\text{uv}}F_1(x_{w-1}), {}_{\text{uv}}F_2(x_w) - {}_{\text{uv}}F_2(x_{w-1}), \dots, {}_{\text{uv}}F_n(x_w) - {}_{\text{uv}}F_n(x_{w-1}))'$.

Proof. Using properties of permanent , it can be written

$$per[\underset{m_1}{{}_{\text{uv}}\mathrm{F}(x_1)}\quad \underset{m_2-m_1}{{}_{\text{uv}}\mathrm{F}(x_2)-{}_{\text{uv}}\mathrm{F}(x_1)}\quad \dots\quad \underset{n-m_d}{1-{}_{\text{uv}}\mathrm{F}(x_d)}]$$

$$=\sum_{t_d=0}^{n-m_d}(-1)^{n-m_d-t_d}\binom{n-m_d}{t_d}\dots\sum_{t_1=0}^{m_2-m_1}(-1)^{m_2-m_1-t_1}\binom{m_2-m_1}{t_1}per[\underset{m_2-t_1}{{}_{\text{uv}}\mathrm{F}(x_1)}\ \dots\ \underset{t_d}{1}\quad \underset{n-m_d-t_d+t_{d-1}}{{}_{\text{uv}}\mathrm{F}(x_d)}]$$

$$=\sum_{t_d=0}^{n-m_d}\dots\sum_{t_1=0}^{m_2-m_1}(-1)^{n-m_1-\sum_{w=1}^{d}t_w}\left[\prod_{w=1}^{d}\binom{m_{w+1}-m_w}{t_w}\right]\sum_{n_s=n-t_d}t_d!per[\underset{m_2-t_1}{{}_{\text{uv}}\mathrm{F}(x_1)}\ \dots\ \underset{n-m_d-t_d+t_{d-1}}{{}_{\text{uv}}\mathrm{F}(x_d)}]\quad [s/.),$$

where $1 = (1, 1, ..., 1)'$. Thus, (10) is obtained.

We will now obtain four expressions for the *df*.

Result 2.1. From(6) - (10), the joint *df* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_d:n}$ can be expressed as

$$
\begin{aligned}
{}_{uv}F_{r_1,r_2,...,r_d:n}(x_1, x_2, ..., x_d) &= \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} C\, per[\underset{m_1}{\Delta_{uv}\mathrm{F}(x_1)} \quad \underset{m_2-m_1}{\Delta_{uv}\mathrm{F}(x_2)} \quad ... \quad \underset{n-m_d}{\Delta_{uv}\mathrm{F}(x_{d+1})}] \\
&= \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} C \sum_P \prod_{w=1}^{d+1} \prod_{l=m_{w-1}+1}^{m_w} \Delta_{uv}F_{j_l}(x_w) \\
&= \sum\sum \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} n!\; C \prod_{w=1}^{d+1} [\Delta_{uv}F^s(x_w)]^{m_w-m_{w-1}} \\
&= \sum\sum \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} n!\, C \left[{}_{uv}F^s(x_1)\right]^{m_1} \prod_{w=2}^{d+1} \sum_{t=m_{w-1}}^{m_w} (-1)^{m_w-t} \binom{m_w - m_{w-1}}{t - m_{w-1}} \left[{}_{uv}F^s(x_w)\right]^{t-m_{w-1}} \left[{}_{uv}F^s(x_{w-1})\right]^{m_w-t} \qquad (11) \\
&= \sum\sum \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} n!\; C \sum_{t_d,...,t_2,t_1}^{n,...,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \prod_{w=1}^{d} \binom{m_{w+1} - m_w}{t_w - m_w} \left[{}_{uv}F^s(x_w)\right]^{n_{s_w}} \\
&= \sum_{m_d,...,m_2,m_1}^{n,...,m_3,m_2} C \sum_{t_d,...,t_2,t_1}^{n,...,m_3,m_2} (-1)^{\sum_{w=1}^{d}(m_{w+1}-t_w)} \left[\prod_{w=1}^{d} \binom{m_{w+1} - m_w}{t_w - m_w}\right] \sum_{n_s=n-t_d+m_d} (t_d - m_d)! \\
&\qquad .per[\underset{m_2-t_1+m_1}{{}_{uv}\mathrm{F}(x_1)} \quad \underset{m_3-m_1-t_2+t_1}{{}_{uv}\mathrm{F}(x_2)} \quad ... \quad \underset{n-m_{d-1}-t_d+t_{d-1}}{{}_{uv}\mathrm{F}(x_d)}] \quad [s/.),
\end{aligned}
$$

where $C = \prod_{w=1}^{d+1} [(m_w - m_{w-1})!]^{-1}$.

We note that $P\{X_{r_1:n} \le x_1, X_{r_2:n} \le x_2, ..., X_{r_d:n} \le x_d\} = \sum\sum P\{X^s_{r_1:n} \le x_1,\ X^s_{r_2:n} \le x_2,\ ... ,X^s_{r_d:n} \le x_d\}$.

We will express the following result for *df* of the r^{th} order statistic from *innid* variables from a truncated distribution.

Result 2.2.

$$
\begin{aligned}
{}_{uv}F_{r:n}(x) &= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} per[\underset{m}{{}_{uv}\mathrm{F}(x)} \quad \underset{n-m}{1-{}_{uv}\mathrm{F}(x)}] \\
&= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{P} \left(\prod_{l=1}^{m} {}_{uv}F_{j_l}(x) \right) \prod_{l=m+1}^{n} [1 - {}_{uv}F_{j_l}(x)] \\
&= \sum \sum \sum_{m=r}^{n} \binom{n}{m} [{}_{uv}F^{s}(x)]^{m} \, [1 - {}_{uv}F^{s}(x)]^{n-m} \qquad (12) \\
&= \sum \sum \sum_{m=r}^{n} \binom{n}{m} \sum_{t=m}^{n} (-1)^{n-t} \binom{n-m}{t-m} \, [{}_{uv}F^{s}(x)]^{n-t+m} \\
&= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{t=m}^{n} (-1)^{n-t} \binom{n-m}{t-m} \sum_{n_s=n-t+m} (t-m)! per[\underset{n-t+m}{{}_{uv}\mathrm{F}(x)}][s/.).
\end{aligned}
$$

Proof. In (11), if $d = 1$, (12) is obtained.

In addition,

$$
\begin{aligned}
{}_{uv}F_{r:n}(x) &= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{P} \left(\prod_{l=1}^{m} {}_{uv}F_{j_l}(x) \right) \prod_{l=m+1}^{n} [1 - {}_{uv}F_{j_l}(x)] \\
&= \sum_{m=r}^{n} \frac{1}{m!(n-m)!} \sum_{P} \left(\prod_{l=1}^{m} {}_{uv}F_{j_l}(x) \right) \sum_{t=m}^{n} (-1)^{t-m} \sum_{n_\tau = t-m} \prod_{l=1}^{t-m} {}_{uv}F_{\tau_l}(x),
\end{aligned}
$$

where $\sum_{n_\tau=t-m}$denotes the sum over all $\binom{n-m}{t-m}$ subsets $\tau = \{\tau_1, \tau_2, ..., \tau_{t-m}\}$ of $\{j_{m+1}, j_{m+2}, ..., j_n\}$.

In Result 2.3 and Result 2.4, the *df's* of minimum and maximum order statistics from *innid* variables from a truncated distribution are given, respectively.

Result 2.3.

$$
\begin{aligned}
{}_{uv}F_{1:n}(x) &= 1 - \frac{1}{n!} per[\underset{n}{1-{}_{uv}\mathrm{F}(x)}] \\
&= 1 - \frac{1}{n!} \sum_{P} \prod_{l=1}^{n} [1 - {}_{uv}F_{j_l}(x)] \\
&= \sum \sum [1 - (1 - {}_{uv}F^{s}(x))^{n}] \qquad (13) \\
&= \sum \sum [1 - \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{t} ({}_{uv}F^{s}(x))^{n-t}] \\
&= 1 - \frac{1}{n!} \sum_{t=0}^{n} (-1)^{n-t} \binom{n}{t} \sum_{n_s=n-t} t! per[\underset{n-t}{{}_{uv}\mathrm{F}(x)}][s/.).
\end{aligned}
$$

Proof. In (12), if $r = 1$, (13) is obtained.

Result 2.4.

$$
\begin{aligned}
{}_{uv}F_{n:n}(x) &= \frac{1}{n!} per[\underset{n}{{}_{uv}\mathrm{F}(x)}] \\
&= \frac{1}{n!} \sum_P \prod_{l=1}^{n} {}_{uv}F_{j_l}(x) \\
&= \sum \sum [{}_{uv}F^s(x)]^n .
\end{aligned} \tag{14}
$$

Proof. In (12), if $r = n$, (14) is obtained.

3. IDENTITIES LEADING TO PROBABILITY DENSITY FUNCTION

The identities in the following theorems will be used to obtain joint *pdf* of order statistics from *innid* variables from a truncated distribution. The identities connect order statistics of *innid* random variables to that of *iid* random variables using (4) and (5). We will now express five theorems used to obtain the *pdf* of order statistics from *innid* variables from a truncated distribution.

Theorem 3.1.

$$
\begin{aligned}
& per[\underset{r_1-1}{\Delta\, {}_{uv}\mathrm{F}(x_1)}\ \underset{1}{{}_{uv}\mathrm{f}(x_1)}\ \underset{r_2-r_1-1}{\Delta\, {}_{uv}\mathrm{F}(x_2)}\ \underset{1}{{}_{uv}\mathrm{f}(x_2)}\ \underset{1}{{}_{uv}\mathrm{f}(x_d)}\ \underset{n-r_d}{\Delta\, {}_{uv}\mathrm{F}(x_{d+1})}] \\
&= \sum_P \left(\prod_{w=1}^{d+1} \prod_{l=r_{w-1}+1}^{r_w-1} \Delta_{uv}F_{j_l}(x_w) \right) \prod_{w=1}^{d} {}_{uv}f_{j_{r_w}}(x_w),
\end{aligned} \tag{15}
$$

where ${}_{uv}\mathrm{f}(x_w) = ({}_{uv}f_1(x_w), {}_{uv}f_2(x_w), \ldots, {}_{uv}f_n(x_w))'$, $r_0 = 0$ and $r_{d+1} = n + 1$.

Proof. Using expansion of permanent, it can be written

$$
\begin{aligned}
& per[\underset{r_1-1}{\Delta\, {}_{uv}\mathrm{F}(x_1)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_1)}\quad \underset{r_2-r_1-1}{\Delta\, {}_{uv}\mathrm{F}(x_2)}\quad \underset{1}{{}_{uv}\mathrm{f}(x_2)}\ \ldots\ \underset{1}{{}_{uv}\mathrm{f}(x_d)}\quad \underset{n-r_d}{\Delta\, {}_{uv}\mathrm{F}(x_{d+1})}] \\
&= \sum_P \Delta_{uv}F_{j_1}(x_1) \ldots \Delta_{uv}F_{j_{r_1-1}}(x_1)\, {}_{uv}f_{j_{r_1}}(x_1)\, \Delta_{uv}F_{j_{r_1+1}}(x_2) \ldots \Delta_{uv}F_{j_{r_2-1}}(x_2)\, {}_{uv}f_{j_{r_2}}(x_2) \ldots {}_{uv}f_{j_{r_d}}(x_d) \\
&\qquad . \Delta_{uv}F_{j_{r_d+1}}(x_{d+1}) \ldots \Delta_{uv}F_{j_n}(x_{d+1}) \\
&= \sum_p \left(\prod_{l=1}^{r_1-1} \Delta_{uv}F_{j_l}(x_1) \right) {}_{uv}f_{j_{r_1}}(x_1) \left(\prod_{l=r_1+1}^{r_2-1} \Delta_{uv}F_{j_l}(x_2) \right) {}_{uv}f_{j_{r_2}}(x_2) \ldots {}_{uv}f_{j_{r_d}}(x_d) \prod_{l=r_d+1}^{n} \Delta_{uv}F_{j_l}(x_{d+1}).
\end{aligned}
$$

Thus, (15) is obtained.

The following theorem connects the *pdf* of order statistics of *innid* random variables to that of order statistics of *iid* random variables.

Theorem 3.2.

$$\sum_{P}\left(\prod_{w=1}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}\Delta_{uv}F_{j_l}(x_w)\right)\prod_{w=1}^{d}{}_{uv}f_{j_{r_w}}(x_w)=\sum\sum n!\left(\prod_{w=1}^{d+1}[\Delta_{uv}F^s(x_w)]^{r_w-r_{w-1}-1}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w). \quad (16)$$

Proof. Omitted.

Identity (16) can be expressed as (17) using binomial expansion.

Theorem 3.3.

$$\begin{aligned}&\sum\sum n!\left(\prod_{w=1}^{d+1}[\Delta_{uv}F^s(x_w)]^{r_w-r_{w-1}-1}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w)\\&=\sum\sum n!\,[{}_{uv}F^s(x_1)]^{r_1-1}\left(\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}}\right.\\&\qquad\left.\cdot[{}_{uv}F^s(x_w)]^{t-r_{w-1}}[{}_{uv}F^s(x_{w-1})]^{r_w-1-t}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w)\,.\end{aligned} \quad (17)$$

Proof. From binomial expansion, (17) is obtained.

Theorem 3.4.

$$\begin{aligned}&\sum\sum n!\,[{}_{uv}F^s(x_1)]^{r_1-1}\left(\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}}[{}_{uv}F^s(x_w)]^{t-r_{w-1}}[{}_{uv}F^s(x_{w-1})]^{r_w-1-t}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w)\\&=\sum\sum n!\sum_{t_d,\ldots,t_2,t_1}^{n,\ldots,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}[{}_{uv}F^s(x_w)]^{n_{s_w}}{}_{uv}f^s(x_w),\end{aligned} \quad (18)$$

where $n_{s_w}=r_{w+1}-r_{w-1}-1-t_w+t_{w-1}$ and $t_0=r_1-1$.

Proof.

$$\begin{aligned}&\sum\sum n!\,[{}_{uv}F^s(x_1)]^{r_1-1}\left(\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}}[{}_{uv}F^s(x_w)]^{t-r_{w-1}}[{}_{uv}F^s(x_{w-1})]^{r_w-1-t}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w)\\&=\sum\sum n!\,[{}_{uv}F^s(x_1)]^{r_1-1}\sum_{t_1=r_1}^{r_2-1}(-1)^{r_2-1-t_1}\binom{r_2-r_1-1}{t_1-r_1}[{}_{uv}F^s(x_2)]^{t_1-r_1}[{}_{uv}F^s(x_1)]^{r_2-1-t_1}\sum_{t_2=r_2}^{r_3-1}(-1)^{r_3-1-t_2}\binom{r_3-r_2-1}{t_2-r_2}\\&\cdot[{}_{uv}F^s(x_3)]^{t_2-r_2}[{}_{uv}F^s(x_2)]^{r_3-1-t_2}\ \ldots\sum_{t_d=r_d}^{n}(-1)^{n-t_d}\binom{n-r_d}{t_d-r_d}[{}_{uv}F^s(x_d)]^{n-t_d}\prod_{w=1}^{d}{}_{uv}f^s(x_w)\\&=\sum\sum n!\sum_{t_d=r_d}^{n}\ldots\sum_{t_2=r_2}^{r_3-1}\sum_{t_1=r_1}^{r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}[{}_{uv}F^s(x_w)]^{n_{s_w}}{}_{uv}f^s(x_w).\end{aligned}$$

Thus, (18) is obtained .

Identity (15) can be expressed as (19) using properties of permanent.

Theorem 3.5.

$$per[\underset{r_1-1}{{}_{uv}F(x_1)} \quad \underset{1}{{}_{uv}f(x_1)} \quad \underset{r_2-r_1-1}{{}_{uv}F(x_2)-{}_{uv}F(x_1)} \quad \underset{1}{{}_{uv}f(x_2)} \ \dots \ \underset{1}{{}_{uv}f(x_d)} \quad \underset{n-r_d}{1-{}_{uv}F(x_d)}]$$

$$= \sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1} (-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)} \left[\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}\right] \sum_{n_S=n+r_d-t_d}(t_d-r_d)! \tag{19}$$

$$.per[\underset{r_2+r_1-2-t_1}{{}_{uv}F(x_1)} \ \dots \ \underset{n-r_{d-1}-t_d+t_{d-1}}{{}_{uv}F(x_d)} \quad \underset{1}{{}_{uv}f(x_1)} \dots \underset{1}{{}_{uv}f(x_d)}][S/.] .$$

Proof. Using properties of permanent, it can be written

$$per[\underset{r_1-1}{{}_{uv}F(x_1)} \quad \underset{1}{{}_{uv}f(x_1)} \quad \underset{r_2-r_1-1}{{}_{uv}F(x_2)-{}_{uv}F(x_1)} \quad \underset{1}{{}_{uv}f(x_2)} \ \dots \ \underset{1}{{}_{uv}f(x_d)} \quad \underset{n-r_d}{1-{}_{uv}F(x_d)}]$$

$$= \sum_{t_d=0}^{n-r_d}(-1)^{n-r_d-t_d}\binom{n-r_d}{t_d} \dots \sum_{t_1=0}^{r_2-r_1-1}(-1)^{r_2-r_1-1-t_1}\binom{r_2-r_1-1}{t_1} per[\underset{r_2-2-t_1}{{}_{uv}F(x_1)} \quad \underset{1}{{}_{uv}f(x_1)} \dots \underset{1}{{}_{uv}f(x_d)} \quad \underset{t_d}{1} \quad \underset{n-r_d-t_d+t_{d-1}}{{}_{uv}F(x_d)}\ ]$$

$$= \sum_{t_d=0}^{n-r_d} \dots \sum_{t_1=0}^{r_2-r_1-1} (-1)^{n+1-r_1-d-\sum_{w=1}^{d} t_w} \left[\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w}\right] \sum_{n_S=n-t_d} t_d!\,per[\underset{r_2-2-t_1}{{}_{uv}F(x_1)} \dots \underset{n-r_d-t_d+t_{d-1}}{{}_{uv}F(x_d)} \quad \underset{1}{{}_{uv}f(x_1)} \dots \underset{1}{{}_{uv}f(x_d)}][S/.].$$

Thus, (19) is obtained.

We will now obtain four expressions for the *pdf*.

Result 3.1. From (15)- (19), the joint *pdf* of $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_d:n}$ can be expressed as

$$f_{r_1,r_2,\dots,r_d:n}(x_1,x_2,\dots,x_d) = D\,per[\underset{r_1-1}{\Delta\,{}_{uv}F(x_1)} \ \underset{1}{{}_{uv}f(x_1)} \ \underset{r_2-r_1-1}{\Delta\,{}_{uv}F(x_2)} \ \underset{1}{{}_{uv}f(x_2)} \ \underset{1}{{}_{uv}f(x_d)} \ \underset{n-r_d}{\Delta\,{}_{uv}F(x_{d+1})}]$$

$$= D\sum_P \left(\prod_{w=1}^{d+1}\prod_{l=r_{w-1}+1}^{r_w-1}\Delta_{uv}F_{j_l}(x_w)\right)\prod_{w=1}^{d}{}_{uv}f_{j_{r_w}}(x_w)$$

$$= \sum\sum n!D\left(\prod_{w=1}^{d+1}[\Delta_{uv}F^s(x_w)]^{r_w-r_{w-1}-1}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w)$$

$$= \sum\sum n!\,D\,[{}_{uv}F^s(x_1)]^{r_1-1}\left(\prod_{w=2}^{d+1}\sum_{t=r_{w-1}}^{r_w-1}(-1)^{r_w-1-t}\binom{r_w-r_{w-1}-1}{t-r_{w-1}}[{}_{uv}F^s(x_w)]^{t-r_{w-1}}[{}_{uv}F^s(x_{w-1})]^{r_w-1-t}\right)\prod_{w=1}^{d}{}_{uv}f^s(x_w) \tag{20}$$

$$= \sum\sum n!\,D\sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\prod_{w=1}^{d}\binom{r_{w+1}-r_w-1}{t_w-r_w}[{}_{uv}F^s(x_w)]^{n_{S_w}}\,{}_{uv}f^s(x_w)$$

$$= D\sum_{t_d,\dots,t_2,t_1}^{n,\dots,r_3-1,r_2-1}(-1)^{-d+\sum_{w=1}^{d}(r_{w+1}-t_w)}\left[\prod_{w=1}^{d}\binom{r_{w+1}r_w-1}{t_w-r_w}\right]\sum_{n_S=n+r_d-t_d}(t_d-r_d)!$$

$$.per[\underset{r_2+r_1-2-t_1}{{}_{uv}F(x_1)} \ \dots \ \underset{n-r_{d-1}-t_d+t_{d-1}}{{}_{uv}F(x_d)} \quad \underset{1}{{}_{uv}f(x_1)} \dots \underset{1}{{}_{uv}f(x_d)}][S/.],$$

where $D = \prod_{w=1}^{d+1}[(r_w-r_{w-1}-1)!]^{-1}$.

By regarding δx_w $(w = 1, 2, ..., d)$ as small, we note that

$$P\{x_1 < X_{r_1:n} \le x_1 + \delta x_1, x_2 < X_{r_2:n} \le x_2 + \delta x_2, ..., x_d < X_{r_d:n} \le x_d + \delta x_d\}$$

$$= \sum\sum P\{x_1 < X^{s}_{r_1:n} \le x_1 + \delta x_1,\ x_2 < X^{s}_{r_2:n} \le x_2 + \delta x_2,\ \ldots, x_d < X^{s}_{r_d:n} \le x_d + \delta x_d\}.$$

We will express the following result for *pdf* of the r^{th} order statistic from *innid* variables from a truncated distribution.

Result 3.2.

$$ {}_{uv}f_{r:n}(x) = \frac{1}{(r-1)!(n-r)!} per[\underset{r-1}{{}_{uv}\mathrm{F}(x)}\ \underset{1}{{}_{uv}\mathrm{f}(x)}\ \underset{n-r}{1 - {}_{uv}\mathrm{F}(x)}]$$

$$= \frac{1}{(r-1)!(n-r)!} \sum_P \left(\prod_{l=1}^{r-1} {}_{uv}F_{j_l}(x)\right)\left(\prod_{l=r+1}^{n}[1 - {}_{uv}F_{j_l}(x)]\right) {}_{uv}f_{j_r}(x)$$

$$= \sum\sum r\binom{n}{r}[{}_{uv}F^{s}(x)]^{r-1}[1 - {}_{uv}F^{s}(x)]^{n-r}\,{}_{uv}f^{s}(x) \tag{21}$$

$$= \sum\sum r\binom{n}{r}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}[{}_{uv}F^{s}(x)]^{n-t+r-1}\,{}_{uv}f^{s}(x)$$

$$= \frac{1}{(r-1)!(n-r)!}\sum_{t=r}^{n}(-1)^{n-t}\binom{n-r}{t-r}\sum_{n_{\mathcal{S}}=n-t+r}(t-r)!\,per[\underset{n-t+r-1}{{}_{uv}\mathrm{F}(x)}\quad \underset{1}{{}_{uv}\mathrm{f}(x)}][\mathcal{S}/.).$$

Proof. In (20), if $d = 1$, (21) is obtained.

In Results 3.3 and 3.4, the *df's* of minimum and maximum order statistics from *innid* variables from a truncated distribution are given, respectively.

Result 3.3.

$$ {}_{uv}f_{1:n}(x) = \frac{1}{(n-1)!} per[\underset{1}{{}_{uv}\mathrm{f}(x)}\ \underset{n-1}{1 - {}_{uv}\mathrm{F}(x)}]$$

$$= \frac{1}{(n-1)!}\sum_P\left(\prod_{l=2}^{n}[1 - {}_{uv}F_{j_l}(x)]\right){}_{uv}f_{j_1}(x)$$

$$= \sum\sum n[1 - {}_{uv}F^{s}(x)]^{n-1}\,{}_{uv}f^{s}(x)$$

$$= \sum\sum n\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}[{}_{uv}F^{s}(x)]^{n-t}\,{}_{uv}f^{s}(x)$$

$$= \frac{1}{(n-1)!}\sum_{t=1}^{n}(-1)^{n-t}\binom{n-1}{t-1}\sum_{n_{S}=n-t+1}(t-1)!\,per[\underset{n-t}{{}_{uv}\mathrm{F}(x)}\quad \underset{1}{{}_{uv}\mathrm{f}(x)}][\mathcal{S}/.). \tag{22}$$

Proof. In (21), if $r = 1$, (22) is obtained.

Result 3.4.

$$
\begin{aligned}
{}_{uv}f_{n:n}(x) &= \frac{1}{(n-1)!} per[{}_{uv}\underset{n-1}{\mathrm{F}(x)} \quad {}_{uv}\underset{1}{\mathrm{f}(x)}] \\
&= \frac{1}{(n-1)!}\sum_P \left(\prod_{l=1}^{n-1} {}_{uv}F_{j_l}(x)\right) {}_{uv}f_{j_n}(x) \\
&= \sum\sum n\,[{}_{uv}F^s(x)]^{n-1}\,{}_{uv}f^s(x).
\end{aligned} \tag{23}
$$

Proof. In (21), if $r = n$, (23) is obtained.

In Result 3.5, the joint *pdf* of $X_{1:n}$ and $X_{n:n}$ is expressed as (24).

Result 3.5.

$$
\begin{aligned}
&{}_{uv}f_{1,n:n}(x_1, x_2) = \frac{1}{(n-2)!} per[{}_{uv}\underset{n-2}{\mathrm{F}(x_2) - {}_{uv}\mathrm{F}(x_1)}\; {}_{uv}\underset{1}{\mathrm{f}(x_1)}\; {}_{uv}\underset{1}{\mathrm{f}(x_2)}] \\
&= \frac{1}{(n-2)!}\sum_P {}_{uv}f_{j_1}(x_1)\left(\prod_{l=2}^{n-1}[{}_{uv}F_{j_l}(x_w) - {}_{uv}F_{j_l}(x_{w-1})]\right){}_{uv}f_{j_n}(x_2) \\
&= \sum\sum n(n-1)[{}_{uv}F^s(x_2) - {}_{uv}F^s(x_1)]^{n-2}\,{}_{uv}f^s(x_1)\,{}_{uv}f^s(x_2) \\
&= \sum\sum n(n-1)\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1}[{}_{uv}F^s(x_1)]^{n-1-t}[{}_{uv}F^s(x_2)]^{t-1}\,{}_{uv}f^s(x_1)\,{}_{uv}f^s(x_2) \\
&= \frac{1}{(n-2)!}\sum_{t=1}^{n-1}(-1)^{n-1-t}\binom{n-2}{t-1} per[\mathrm{uv}\underset{n-1-t}{\mathrm{F}(x_1)} \quad \mathrm{uv}\underset{t-1}{\mathrm{F}(x_2)} \quad \mathrm{uv}\underset{1}{\mathrm{f}(x_1)} \quad \mathrm{uv}\underset{1}{\mathrm{f}(x_2)}].
\end{aligned} \tag{24}
$$

Proof. In (20), if $d = 2$ and $r_1 = 1$, $r_2 = n$, (24) is obtained.

In the following result, we will give the joint *pdf* of $X_{1:n}, X_{2:n}, \ldots, X_{k\cdot n}$.

Result 3.6.

$$
\begin{aligned}
&{}_{uv}f_{1,2,\ldots,k:n}(x_1,x_2,\ldots,x_\mathrm{k}) = \frac{1}{(n-k)!} per[\underset{n-k}{1 - {}_{uv}\mathrm{F}(x_k)}\;\; {}_{uv}\underset{1}{\mathrm{f}(x_1)}\;\; {}_{uv}\underset{1}{\mathrm{f}(x_2)} \ldots {}_{uv}\underset{1}{\mathrm{f}(x_k)}] \\
&= \frac{1}{(n-k)!}\sum_P\left(\prod_{l=k+1}^{n}[1 - {}_{uv}F_{j_l}(x_k)]\right){}_{uv}f_{j_1}(x_1)\,{}_{uv}f_{j_2}(x_2)\ldots{}_{uv}f_{j_k}(x_k) \\
&= \sum\sum\frac{n!}{(n-k)!}[1 - {}_{uv}F^s(x_k)]^{n-k}\prod_{w=1}^{k}{}_{uv}f^s(x_w) \\
&= \sum\sum\frac{n!}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\binom{n-k}{t-k}[{}_{uv}F^s(x_k)]^{n-t}\prod_{w=1}^{k}{}_{uv}f^s(x_w) \\
&= \frac{1}{(n-k)!}\sum_{t=k}^{n}(-1)^{n-t}\binom{n-k}{t-k}\sum_{n_S=n-t+k}(t-k)!\,per[{}_{uv}\underset{n-t}{\mathrm{F}(x_k)}\;\; {}_{uv}\underset{1}{\mathrm{f}(x_1)}\;\; {}_{uv}\underset{1}{\mathrm{f}(x_2)}\ldots\;\; {}_{uv}\underset{1}{\mathrm{f}(x_k)}][\mathcal{S}/.).
\end{aligned} \tag{25}
$$

Proof. In (20), if $d = k$ and $r_1 = 1$, $r_2 = 2, \ldots, r_k = k$, (25) is obtained

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متطابقات تعطى التوزيعات المشتركة للإحصاءات المرتبة لمتغيرات ***innid*** من توزيع مقسم

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خلاصة

تهتم هذه الدراسة بالتوزيعات المشتركة للإحصاءات المرتبة من متغيرات innid من توزيعات مقسمة وذلك بدلالة متطابقات متخصصة ومن ثم نعطي بعض النتائج التي تربط بين توزيعات الإحصاءات المرتبة من متغيرات innid من توزيع مقسم وبين توزيعات مرتبة لمتغيرات مستقلة تتبع نفس التوزيع الاحتمالي.

كلمات معبرة: متغير عشوائي innid – توزيع مشترك – توزيع مقسم – إحصاءات مرتبة.