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Common fixed point theorems for six mappings with some weaker conditions in 2-metric spaces

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ABSTRACT

In this paper, we prove unique common fixed-point theorems for six mappings (two set-valued and four single-valued mappings) without assuming compatibility and continuity of any mapping on noncomplete 2-metric spaces. To prove these theorems, we introduce a noncompatible condition, that is, weak commutativity of type (Kh) in a 2-metric space. We show that completeness of the whole space is not necessary for the existence and uniqueness of common fixed point. Also, we prove common fixed point theorems for two self mappings and two sequences set-valued mappings by the same weaker conditions. Our results improve, extend and generalize the corresponding results given by many authors.

Keywords: Common fixed point; single and set-valued mappings; weak commutativity of type (Kh).

INTRODUCTION

Fixed point theorems for hybrid pair of set and single valued mappings have numerous applications in science and engineering (e.g. Abd El-Monsef *et al.*, 2007, 2009; Abu-Donia & Abd-Rabou, 2009, 2010; Border, 1990). In recent years several fixed point theorems for single and set-valued maps for pairs of mappings have numerous applications and by now there exists an extensive considerable and rich literature in this domain. Those common fixed-point theorems for single and set-valued maps are interesting and play a major role in many areas. The concept of a 2-metric space is a natural generalization of a metric space. It has been investigated initially by Gähler (1963) and has been developed extensively by Gähler (1965,1966) and many others. The topology induced by 2-metric space is called 2-metric topology, which is generated by the set of all open spheres with two centers. Many authors used topology in many applications; for example, El Naschie (2002, 2006) used this sort of topology in physical applications, Abu-Donia & Abd-Rabou (2009, 2010) studied common fixed-point theorems for single- and set-valued mappings in fuzzy 2-metric spaces. Iseki (1975) and others (Hsiao, 1986; Naidu, 2001; Pathak *et al., 1995; Popa et al.*, 2010) studied the fixed point theorems in 2-metric spaces. Singh (1979) studied some contractive type principles in 2-metric spaces and applications. Abd EL-Monsef *et al.* (2007, 2009) generalized some definitions on 2-metric spaces and studied common fixed-point theorems for single and set-valued mappings in 2-metric spaces. In this paper, we introduce a new noncompatible condition, that is, weak commutativity of type (Kh) in a 2-metric space. We prove common fixed point theorems for hybrid pairs of set and single-valued mappings by using a noncompatible condition, i.e., weak commutativity of type (Kh) in 2-metric spaces.These theorems generalize, extend and improve the corresponding results given by many authors.

BASIC PRELIMINARIES

Definition 2.1 (Gähler, 1963). Let X denote a nonempty set and $R$the set of all non-negative number. Then X together with a function $d: X \times X \times X \rightarrow R$ is called a 2-metric space if it satisfies the following properties:

(1) for distinct points $x, y \in X$, there exists a point $c \in X$ such that $d(x, y, c) \neq 0$ and $d(x, y, c) = 0$ if at least two of x, y and c are equal,

(2) $d(x, y, c) = d(x, c, y) = d(y, x, c) = d(y, c, x) = d(c, x, y) = d(c, y, x)$ (Symmetry),

(3) $d(x, y, c) \leq d(x, y, z) + d(x, z, c) + d(z, y, c)$ for $x, y, c, z \in X$ (Rectangle inequality).

The function d is called a 2-metric for the space X and the pair (X, d) denotes 2-metric space. It was shown by Gähler (1965) that 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. A 2-metric d, which is continuous in all of its arguments is said to be continuous.

Geometrically, the value of a 2-metric $d(x, y, c)$ represents the area of a triangle with vertices x, y and c.

Definition 2.2 (Pathak *et al.*, 1995). A sequence $\{y_n\}$ in a 2-metric space (X, d) is said to be convergent to a point $y \in X$, denoted by $\lim_{n\to\infty} y_n = y$ if $\lim_{n\to\infty} d(y_n, y, c) = 0$ for all $c \in X$. The point y is called the limit of the sequence $\{y_n\}$ in X.

Definition 2.3 (Pathak *et al.*, 1995). A sequence $\{y_n\}$ in a 2-metric space (X, d) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} d(y_m, y_n, c) = 0$ for all $c \in X$.

Definition 2.4 (Pathak *et al.*, 1995). A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point of X.

Remark 2.1 We note that, in a metric space a convergent sequence is a Cauchy sequence and in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X (Naidu & Prasad 1986).

Throughout this paper d stands for a continuous d-metric. In the sequel, (X, d) denotes a 2-metric space and $B(X)$ is the set of all nonempty bounded subsets of X. As in Abd El-Monsef *et al.* (2007, 2009), we define.

$$\delta(A, B, C) = sup\{d(a, b, c) : a \in A, b \in B, c \in C\},$$
$$D(A, B, C) = inf\{d(a, b, c) : a \in A, b \in B, c \in C\}.$$

If $A = \{a\}$ we denote $\delta(a, B, C), D(a, B, C)$ for $\delta(A, B, C)$ and $D(A, B, C)$ respectively. If $A = \{a\}, B = \{b\}$ and $C = \{c\}$, one can deduce that $\delta(A, B, C) = D(A, B, C) = d(a, b, c)$. It follows immediately from the definition of $\delta(A, B, C)$ that

$$\delta(A, B, C) = \delta(B, A, C) = ... = \delta(C, B, A) \geq 0, \quad \delta(A, B, C) = 0,$$

iff at least two of A,B,C consist of equal single points,

$$\delta(A, B, C) \leq \delta(A, B, E) + \delta(A, E, C) + \delta(E, B, C), \text{for all } A, B, C, E \in B(X).$$

Definition 2.5 (Abd EL-Monsef *et al.*, 2007, 2009). A sequence $\{A_n\}$ of nonempty subset of a 2-metric space (X, d) is said to be convergent to a subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\} \in X$ such that $a_n \in A_n$ for all $n \in N$ and $\lim_{n\to\infty} d(a_n, a, c) = 0$,

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Lemma 2.1 (Abd El-Monsef *et al.*, 2007, 2009). If $\{A_n\}$ and $\{B_n\}$ are sequence in $B(X)$ converging to A and B respectively in $B(X)$, then the sequence $\{\delta(A_n, B_n, C\}$ converges to $\delta(A, B, C)$ for $C \in B(X)$.

Lemma 2.2 (Abd El-Monsef *et al.*, 2007, 2009). Let$\{A_n\}$ be a sequence in $B(X)$ and $y \in X$ such that $\delta(A_n, y, C) \to 0$. Then the sequence $\{A_n\}$ converging to the set$\{y\}$ in $B(X)$.

Definition 2.6 (Abd El-Monsef *et al.*, 2007, 2009). The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly commuting if $C, IFx \in B(X)$ and

$\delta(FIx, IFx, C) \leq max\{\delta(Ix, Fx, C), \delta(IFx, IFx, C)\}$ for all $x \in X$.

Note that, if F is a single-valued mapping, then the set $\{IFx\}$ consists of a single point. Therefore, $\delta(IFx, IFx, C) = d(IFx, IFx, C) = 0$ and above inequality reduces to the well known condition given by Khan (1984), that is $d(FIx, IFx, c) \leq d(Ix, Fx, c)$.

Definition 2.7 (Abd El-Monsef *et al.*, 2007, 2009). The mappings $I : X \to X$ and $F : X \to B(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X), Fx_n \to \{t\}, Ix_n \to t$, for some t in X.

Definition 2.8 The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points. i.e. for each point $u \in X$ such that $Iu \in Fu$, we have $FIu = IFu$. Note that the equation $Fu = \{Iu\}$ implies that Fu is singleton.

Every δ-compatible pair of hybrid maps is weakly compatible but the converse is false (Abd El-Monsef *et al.*, 2007, 2009). The notion of weakly commuting hybrid pair of type(KB) in metric spaces is introduced in Kubiaczyk & Deshpande (2008).

Definition 2.9 (Kubiaczyk & Deshpande, 1985). The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting of type (KB) at x in metric space if there exists some positive real number R such that $\delta(IIx, FIx) \leq R\delta(Ix, Fx)$. Here I and F are weakly commuting of type (KB) on X if the above inequality holds for all x.

In this paper, we extend the notion of weakly commuting hybrid pair into 2-metric spaces and it name weakly commuting of type (Kh) as the following:

Definition 2.10 The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting of type (Kh) at $x \in X$ if there exists some positive real number R such that $\delta(IIx, FIx, C) \leq R\delta(Ix, Fx, C), C \in B(X)$. Here I and F are weakly commuting of type (Kh) on X if the above inequality holds for all $x \in X$.

Remark 2.2 Every weakly compatible pair of hybrid maps is weakly commuting of type (Kh) but the converse is not necessarily true.

Example 2.1 Let $X = [1, 10]$. Define $I : X \to X$ and $F : X \to B(X)$ by

$$Ix = \begin{cases} x & : \quad 1 \leq x \leq 5 \\ \frac{x+3}{4} & : \quad 5 < x \leq 10 \end{cases}, Fx = \begin{cases} [1, x] & : \quad 1 \leq x \leq 2 \\ [2, x] & : \quad 2 < x \leq 5 \\ \left[2, \dfrac{x+1}{3}\right] & : \quad 5 < x \leq 10 \end{cases}$$

$\delta(A, B, C) = max\{d(a, b, c) : a \in A, b \in B, c \in C\}, A, B, C \in B(X)$,

where $d(a, b, c) = max\{|x - y|, |y - z|, |z - x|\}$.

Let $x_n = 5 + \frac{1}{n}, n = 1, 2, \ldots$. Then,

$$\lim_{n\to\infty} Ix_n = 2 \quad \text{and} \quad \lim_{n\to\infty} Fx_n = \{2\}.$$

Also,

$IFx_n \in B(X)$ and $\delta(FIx_n, IFx_n, C) = \delta([2, 2 + \frac{1}{4n}], [2, 2 + \frac{1}{3n}], C) \to 0$, as $n \to \infty$.

Hence, I and F are δ-compatible and hence weakly compatible. On the other hand, if we take $x = 2$, then $IIx = 2$, $FIx = [1, 2]$ and clearly I and F are weakly commuting of type (Kh) for $x = 2$.

Example 2.2 Let $X = [1, \infty)$. Define $I : X \to X$ and $F : X \to B(X)$ by $Ix = 2x$ and $Fx = [1, x]$ for all $x \in X$,

$\delta(A, B, C) = max\{d(a, b, c) : a \in A, b \in B, c \in C\}, A, B, C \in B(X)$,

where $d(a, b, c) = max\{|x - y|, |y - z|, |z - x|\}$. Then $IIx = 4x, FIx = [1, 2x]$ and for $R > 3$ we can see that $\delta(IIx, FIx, C) < R\delta(Ix, Fx, C)$ for all $x \in X$. Thus I and F are weakly commuting of type (Kh) on X, but there exists no sequence x_n in X such that condition of δ-compatibility is satisfied.

Example 2.3 Let $X = [1, \infty)$. Define $I : X \to X$ and $F : X \to B(X)$ by $Ix = 2x$ and $Fx = [1, x + 1]$ for all $x \in X$,

$\delta(A, B, C) = max\{d(a, b, c) : a \in A, b \in B, c \in C\}, A, B, C \in B(X)$,

where $d(a, b, c) = max\{|x - y|, |y - z|, |z - x|\}$. Then $IIx = 4x, FIx = [1, 2x + 1]$ and for $R > 3$ we can see that $\delta(IIx, FIx, C) < R\delta(Ix, Fx, C)$ for all $x \in X$. Thus I and F are weakly commuting of type (Kh) on X. On the other hand, if we take $x = 1$, thus $I(1) = 2 \in F(1) = [1, 2]$, $IF(1) \neq FI(1)$. Then I and F are not weakly compatible.

Definition 2.11 Let $F : X \to B(X)$ be

(a) A point $x \in X$ is a fixed point for F if x is in Fx,

(b) A point $x \in X$ is a strict fixed point for F if $\{x\} = Fx$.

Kubiaczyk & Deshparde (2008) proved the following theorem:

Theorem 2.1 Let S and T be self mappings of a metric space (X, d) and $A, B : X \to B(X)$ set-valued mappings satisfying following conditions:

(i) $\bigcup A(X) \subseteq S(X)$ and $\bigcup B(X) \subseteq T(X)$,

(ii) the pairs $\{A, T\}$ and $\{B, S\}$ are weakly commuting of type (KB) at coincidence points in X,

(iii) $\delta(Ax, By) \le max\{cd(Tx, Sy), c\delta(Tx, Ax), c\delta(Sy, By), aD(Tx, By) + bD(Sy, Ax)\}$, for all $x, y \in X$, where $0 \le c < 1, a, b \ge 0, a+b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1$. Suppose that one of the mappings $S(X)$ and $T(X)$ is complete subspace of X. Then A, B, S and T have a unique common fixed point.

In this paper, we improve, extend and generalize the corresponding results given by many authors.

MAIN RESULTS

Let S, R, H and T be four self mappings of a 2-metric space (X, d) and $A, B : X \to B(X)$ set-valued mappings satisfying following conditions:

(1) $\bigcup A(X) \subseteq SR(X)$ and $\bigcup B(X) \subseteq TH(X)$,

(2) $\delta(Ax, By, C) \le max\{c\delta(THx, SRy, C), c\delta(THx, Ax, C), c\delta(SRy, By, C),$

$$aD(THx, By, C) + bD(SRy, Ax, C)\},$$

for all $x, y \in X, C \in B(X)$, where $0 \le c < 1, a, b \ge 0, a+b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1$.

Let $x_0 \in X$ be an arbitrary point in X. By (1), there exists a point $x_1 \in X$ such that $SRx_1 \in Ax_0 = Z_0$ and for this point x_1 there exists a point $x_2 \in X$ such that $THx_2 \in Bx_1 = Z_1$ and so on. Continuing in this manner, we can define a sequence as follows:

(3) $SRx_{2n+1} \in Ax_{2n} = Z_{2n}, THx_{2n+2} \in Bx_{2n+1} = Z_{2n+1}, \forall \quad n = 0, 1, 2, ...$

Now we are ready to prove the following lemma for our theorem:

Lemma 3.1 Let S, R, H and T be four self mappings of a 2-metric space (X, d) and $A, B : X \to B(X)$ set-valued mappings satisfying conditions(1) and (2). Then for every $n \in N$ we have $\lim_{n\to\infty} \delta(Z_n, Z_{n+1}, Z_{n+2}) = 0$.

Proof. Since

$$\begin{aligned}\delta(Z_{2n+2}, Z_{2n+1}, Z_{2n}) &= \delta(Ax_{2n+2}, Bx_{2n+1}, Z_{2n})\\ &\le max\{c\delta(THx_{2n+2}, SRx_{2n+1}, Z_{2n}), c\delta(THx_{2n+2}, Ax_{2n+2}, Z_{2n}), c\delta(SRx_{2n+1}, Bx_{2n+1}, Z_{2n}),\\ &\qquad aD(THx_{2n+2}, Bx_{2n+1}, Z_{2n}) + bD(SRx_{2n+1}, Ax_{2n+2}, Z_{2n})\}\\ &\le max\{c\delta(Z_{2n+1}, Z_{2n}, Z_{2n}), c\delta(Z_{2n+1}, Z_{2n+2}, Z_{2n}), c\delta(Z_{2n}, Z_{2n+1}, Z_{2n}),\\ &\qquad aD(Z_{2n+1}, Z_{2n+1}, Z_{2n}) + bD(Z_{2n}, Z_{2n+2}, Z_{2n})\}\\ &\le c\delta(Z_{2n+2}, Z_{2n+1}, Z_{2n}).\end{aligned}$$

Since $0 \leq c < 1$, we have $\delta(Z_{2n+2}, Z_{2n+1}, Z_{2n}) \to 0$, as $n \to \infty$. Similarly, we have $\delta(Z_{2n+3}, Z_{2n+2}, Z_{2n+1}) \to 0$, as $n \to \infty$. Hence we conclude that $\lim_{n\to\infty} \delta(Z_n, Z_{n+1}, Z_{n+2}) = 0$.

Now we can introduce our main theorems.

Theorem 3.1 Let S, R, H and T be four self mappings of a 2-metric space (X, d) and $A, B : X \to B(X)$ set-valued mappings satisfying the conditions (1), (2) and the pairs $\{A, TH\}$ and $\{B, SR\}$ are weakly commuting of type (Kh) at coincidence points in X. Suppose that one of the mappings $SR(X)$ and $TH(X)$ is complete subspace of X. Then A, B, S, H, R and T have a unique common fixed point, which is a strict fixed point for A and B.

Proof. For simplicity, we put $V_n = \delta(Z_n, Z_{n+1}, C)$ for $n = 0, 1, 2, ...$ By (2) and (3), we have

$$\begin{aligned}
V_{2n} &= \delta(Z_{2n}, Z_{2n+1}, C) = \delta(Ax_{2n}, Bx_{2n+1}, C)\\
&\leq max\{c\delta(THx_{2n}, SRx_{2n+1}, C), c\delta(THx_{2n}, Ax_{2n}, C), c\delta(SRx_{2n+1}, Bx_{2n+1}, C),\\
&\qquad aD(THx_{2n}, Bx_{2n+1}, C) + bD(SRx_{2n+1}, Ax_{2n}, C)\}\\
&\leq max\{cd(Z_{2n-1}, Z_{2n}, C), c\delta(Z_{2n-1}, Z_{2n}, C), c\delta(Z_{2n}, Z_{2n+1}, C),\\
&\qquad aD(Z_{2n-1}, Z_{2n+1}, C) + bD(Z_{2n}, Z_{2n}, C)\}\\
&\leq max\{cV_{2n-1}, cV_{2n}, a(V_{2n-1} + V_{2n})\}\\
&\leq max\{c, \frac{a}{1-a}\} V_{2n-1} \quad \text{for } n \in N.
\end{aligned}$$

Similarly, one can show that

$$V_{2n+1} \leq max\{c, \frac{b}{1-b}\} V_{2n} \quad \text{for } n \in N.$$

If we put

$$\beta = max\{c, \frac{b}{1-b}\}.max\{c, \frac{a}{1-a}\},$$

then by hypothesis it can be easily seen that $0 \leq \beta < 1$. So we deduce that

$$V_{2n} \leq \beta V_{2n-2} \leq ... \leq \beta^n V_0, V_{2n+1} \leq \beta V_{2n-1} \leq ... \leq \beta^n V_1 \quad \text{for } n \in N.$$

Put $M = max\{V_0, V_1\}$. It follows from the above inequality that if z_n is an arbitrary point in the set Z_n for $n \in N$, then we obtain that

$$\delta(z_{2n}, z_{2n+1}, C) \leq \delta(Z_{2n}, Z_{2n+1}, C) \leq \beta^n M,$$

$$\delta(z_{2n+1}, z_{2n+2}, C) \leq \delta(Z_{2n+1}, Z_{2n+2}, C) \leq \beta^n M.$$

This implies that $\{z_n\}$ and any subsequence is a Cauchy sequence in X.

Now suppose that $SR(X)$ is complete:

$$\delta(SRx_{2m+1}, SRx_{2n+1}, C) \leq \delta(Z_{2m}, Z_{2n}, C) \quad \text{for } m, n > n_0, n_0 = 1, 2, 3....$$

For $m > n$, we get

$$\delta(Z_{2m}, Z_{2n}, C) \leq \delta(Z_{2n}, Z_{2n+1}, Z_{2n+2}) + \delta(Z_{2n+1}, Z_{2n+2}, Z_{2n+3}) + ...$$
$$+\delta(Z_{2m-2}, Z_{2m-1}, Z_{2m}) + \delta(Z_{2m-1}, Z_{2m}, C).$$

By taking the limit as $m, n \to \infty$ and using Lemma 3.1, we obtain $\delta(Z_{2m}, Z_{2n}, C) \to 0$. Therefore $\{SRx_{2n+1}\}$ is a Cauchy sequence and hence $\{SRx_{2n+1}\} \to z = SRv \in SR(X)$. But $THx_{2n} \in Bx_{2n-1} = Z_{2n-1}$ and whence we have

$$\delta(THx_{2n}, SRx_{2n+1}, C) \leq \delta(Z_{2n-1}, Z_{2n}, C) = V_{2n-1} \to 0.$$

Consequently, $TH_{2n} \to z$. Moreover, we have for $n = 1, 2, 3...$

$$\delta(Ax_{2n}, z, C) \leq \delta(THx_{2n}, z, C) + \delta(Ax_{2n}, THx_{2n}, C) + \delta(Ax_{2n}, z, THx_{2n}).$$

Therefore, $\delta(Ax_{2n}, z, C) \to 0$. Similarly, it follows that $\delta(Bx_{2n}, z, C) \to 0$.

By (2), we have for $n = 1, 2, 3....$

$$\delta(Ax_{2n}, Bv, C) \leq max\{cd(THx_{2n}, SRv, C), c\delta(THx_{2n}, Ax_{2n}, C), c\delta(SRv, Bv, C),$$
$$aD(THx_{2n}, Bv, C) + bD(SRv, Ax_{2n}, C)\}.$$

Since $\delta(THx_{2n}, Bv, C) \to \delta(z, Bv, C)$, when $THx_{2n} \to z$, we get as $n \to \infty$

$$\delta(z, Bv, C) \leq max\{c, a\}\delta(z, Bv, C),$$

which is a contradiction. Thus $Bv = \{z\} = \{SRv\}$. But $\bigcup_B(X) \subseteq TH(X)\cup$, there exists $u \in X$ such that $\{THu\} = Bv = \{z\} = \{SRv\}$. Now if $Au \neq Bv, \delta(Au, Bv, C) \neq 0$, then by (2), we obtain

$$\delta(Au,Bv,C) \le max\{c\delta(THu,SRv,C), c\delta(THu,Au,C), c\delta(SRv,Bv,C),$$
$$aD(THu,Bv,C) + bD(SRv,Au,C)\}.$$

As $n \to \infty$, we have$\delta(Au,z,C) \le max\{c,b\}\delta(Au,z,C)$. This is a contradiction. Thus we have $Au = \{THu\} = Bv = \{z\} = \{SRv\}$. Since $Au = \{THu\} = \{z\}$ and the pair $\{A, TH\}$ is weakly commuting of type (Kh) at coincidence points in X, we obtain

$\delta(THTHu, ATHu, C) \le R\delta(THu, Au, C)$ which gives $Az = \{THz\}$.

By (2), we get

$$\delta(Az,z,C) \le \delta(Az,Bv,C)$$
$$\le max\{cd(THz,SRv,C), c\delta(THz,Az,C), c\delta(SRv,Bv,C), aD(THz,Bv,C) + bD(SRv,Az,C)\}$$
$$\le max\{c,a+b\}\delta(Az,z,C).$$

Here we reach a contradiction. Thus $Az = \{z\}$. Consequently, we have$Az = \{z\} = \{THz\}$. Similarly $Bz = \{z\} = \{SRz\}$. Therefore, we have $Az = \{THz\} = \{z\} = Bz = \{SRz\}$.

Now, we prove that $Rz = z$. In fact, by (2), it follows that

$$\delta(Az,BRz,C) \le max\{cd(THz,SRRz,C), c\delta(THz,Az,C), c\delta(SRRz,BRz,C),$$
$$aD(THz,BRz,C) + bD(SRRz,Az,C)\}.$$

Since $Bz = \{z\} = \{SRz\}$ and $R: X \to X$, thus $BRz = \{Rz\}, SRRz = Rz$. Then, the above inequality becomes $d(z,Rz,C) \le max\{c,a+b\}d(z,Rz,C)$. This is a contradiction. Thus we have $Rz = z$. Hence $Sz = SRz = z$. Similarly, we get $Tz = Hz = z$. Thus

$$Az = \{Tz\} = \{Hz\} = \{z\} = \{Sz\} = \{Rz\} = Bz.$$

To prove uniqueness, let p be another common fixed point of A, B, S, H, R and T. Then

$$\delta(z,p,C) \le \delta(Az,Bp,C)$$
$$\le max\{cd(THz,SRp,C), c\delta(THz,Az,C), c\delta(SRp,Bp,C), aD(THz,Bp,C) + bD(SRp,Az,C)\}$$
$$\le max\{c,a+b\}d(z,p,C),$$

which is a contradiction, therefore $z = p$. Then A, B, S, H, R and T have a unique common fixed point.

If we put $SR = S$ and $TH = T$ in Theorem 3.1, we get the following:

Theorem 3.2 Let S and T be self mappings of a 2-metric space (X, d) and $A, B : X \to B(X)$ set-valued mappings satisfying following conditions:

(1) $\bigcup A(X) \subseteq S(X)$ and $\bigcup B(X) \subseteq T(X)$,

(2) the pairs $\{A, T\}$ and $\{B, S\}$ are weakly commuting of type (Kh) at coincidence points in X,

(3) $$\delta(Ax, By, C) \leq max\{c\delta(Tx, Sy, C), c\delta(Tx, Ax, C), c\delta(Sy, By, C), aD(Tx, By, C) + bD(Sy, Ax, C)\},$$

for all $x, y \in X, C \in B(X)$, where

$$0 \leq c < 1, a, b \geq 0, a + b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1.$$

Suppose that one of the mappings $S(X)$ and $T(X)$ is complete subspace of X. Then A, B, S and T have a unique common fixed point, which is a strict fixed point for A and B.

Remark 3.1 Theorem 3.2 improves and generalizes the results of Abd El-Monsef *et al.* (2007,2009).

Remark 3.2 Theorem 3.2 extends, improves and generalizes the results of Kubiaczyk & Deshpande (2008) in 2-metric space.

Remark 3.3 From condition (3) in Theorem 3.2, we obtain

$$\begin{aligned} aD(Tx, By, C) \quad &+ bD(Sy, Ax, C) \leq max\{a, b\}(D(Tx, By, C) + D(Sy, Ax, C)) \\ &= max\{2a, 2b\}(\frac{D(Tx, By, C) + D(Sy, Ax, C)}{2}). \end{aligned}$$

Then, condition(3) becomes

$$\begin{aligned} \delta(Ax, By, C) \leq kmax\{&\delta(Tx, Sy, C), \delta(Tx, Ax, C), \delta(Sy, By, C) \\ &, (\frac{D(Tx, By, C) + D(Sy, Ax, C)}{2})\}, \end{aligned}$$

where $k = max\{c, 2a, 2b\} < 1$. Then, Theorem 3.2 improves the results of Corollary 4.1(a) (Popa *et al.*, 2010) and Altun & Turkoglu, 2008) in 2-metric spaces.

Remark 3.4 If we put $A = B$ and $S = T$ in Theorem 3.2, we get extension, improvement and generalization for the results of Imad *et al.* (1988), Imad & Ahmad (1994) and Sessa *et al.* (1986) in 2-metric spaces.

If we put $A = B$ and $SR = TH = S$ in Theorem 3.1, we get the following:

Theorem 3.3 Let S be a self mapping of a 2-metric space (X, d) and $A : X \to B(X)$ a set-valued mapping satisfying following conditions:

(1) $\bigcup A(X) \subseteq S(X)$,

(2) the pair $\{A, S\}$ is weakly commuting of type (Kh) at coincidence points in X,

(3) $$\delta(Ax, Ay, C) \leq max\{c\delta(Sx, Sy, C), c\delta(Sx, Ax, C), c\delta(Sy, Ay, C), aD(Sx, Ay, C) + bD(Sy, Ax, C)\},$$

for all $x, y \in X, C \in B(X)$, where

$$0 \leq c < 1, a, b \geq 0, a + b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1.$$

Suppose that $S(X)$ is complete subspace of X. Then A and S have a unique common fixed point, which is a strict fixed point for A.

Remark 3.5 Theorem 3.3 improves,extensions and generalizes the results of Iseki (1975), Naidu (2001) and Naidu & Prasad (1986) in 2-metric spaces.

Theorem 3.4 Let S be a self mapping of a 2-metric space (X, d) and $A : X \to B(X)$ a set-valued mapping satisfying following conditions:

(1) $\bigcup A^n(X) \subseteq S^m(X)$,

(2) the pairs $\{A^n, S^m\}$ are weakly commuting of type (Kh) at coincidence points in X,

(3) $$\delta(A^n x, A^n y, C) \leq max\{c\delta(S^m x, S^m y, C), c\delta(S^m x, A^n x, C), c\delta(S^m y, A^n y, C), aD(S^m x, A^n y, C) + bD(S^m y, A^n x, C)\},$$

for all $x, y \in X, C \in B(X)$, where

$$0 \leq c < 1, a, b \geq 0, a + b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1.$$

Suppose that one of the mappings $S^m(X)$ is complete subspace of X. Then A and S have a unique common fixed point, which is a strict fixed point for A.

Proof. If we set $A = B = A^n$ and $S = T = S^m$ in Theorem 3.1 A^n and S^m have a unique common fixed point in X. That is, there exists $z \in X$ such that $A^n(z) = \{S^m(z)\} = \{z\}$. since $A^n(Az) = A(A^n z) = Az$, it follows that Az is a fixed point of A^n and S^m and hence $Az = z$. Similarly, we have $Sz = z$.

Theorem 3.5 Let S and T be two self mappings of a 2-metric space (X, d) and two sequences set-valued mappings $A_i, B_j : X \to B(X)$ for all $i, j \in N$ satisfying following conditions:

(1) there exists $i_0, j_0 \in N$ such that $\bigcup A_{i_0}(X) \subseteq S(X)$ and $\bigcup B_{j_0}(X) \subseteq T(X)$,

(2) the pairs $\{A_{i_0}, T\}$ and $\{B_{j_0}, S\}$ are weakly commuting of type (Kh) at coincidence points in X,

(3) $$\delta(A_ix, B_jy, C) \leq max\{c\delta(Tx, Sy, C), c\delta(Tx, A_ix, C), c\delta(Sy, B_jy, C), aD(Tx, B_jy, C) + bD(Sy, A_ix, C)\},$$

for all $x, y \in X$, where

$$0 \leq c < 1, a, b \geq 0, a + b < 1, c \quad max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1 \quad ,$$

and if one of the mappings $S(X)$ and $T(X)$ is complete subspace of X. Then A_i, B_j, S and T have a unique common fixed point for all $i, j = 1, 2, ...,$ which is a strict fixed point for A_i, B_j.

Proof. By Theorem 3.1, the mappings A_{i_0}, B_{j_0}, S and T for some $i_0, j_0 \in N$ have a unique common fixed point in X. That is, there exists a unique point $z \in X$ such that $\{Sz\} = \{Tz\} = \{z\} = A_{i_0}z = B_{j_0}z$.

Suppose that there exists $i \in N$ such that $i \neq i_0$. Then, we have

$$\delta(A_iz, z, C) = \delta(A_iz, B_{j_0}z, C)$$

$$\leq max\{c\delta(Tz, Sz, C), c\delta(Tz, A_iz, C), c\delta(Sz, B_{j_0}z, C), aD(Tz, B_{j_0}z, C) + bD(Sz, A_iz, C)\}$$

$$\leq max\{c, b\}\delta(A_iz, z, C),$$

which is a contradiction. Hence, for all $i \in N$, it follows that $A_iz = z$. Similarly, for all $j \in N$, we have $B_jz = z$. Therefore, for all $i, j \in N$, we have

$$A_iz = B_jz = z = \{Sz\} = \{Tz\}.$$

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مبرهنات نقطة ثابتة مشتركة لست تطبيقات في فضاءات مقاسية بعديتها إثنان وبشروط مخففة

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خلاصة

في هذا البحث نثبت مبرهنات نقطة ثابتة وحيدة مشتركة لست تطبيقات على فضاءات مقاسية غير تامة بعديتها إثنان، وذلك من دون أن نفترض الانسجام او الاستمرارية لأي من التطبيقات. ولإثبات هذه المبرهنات، نستخدم شرطاً غير انسجامي وهو الانسجامية الضعيفة. ونثبت بأن تمام الفضاء بأكمله غير ضروري لوجود ووحدانية النقطة الثاتبة المشتركة. ثم نستخدم شروطنا المخففة لإثبات مبرهنات نقطة ثابتة لتطبيقين ذاتيين ومتتاليين. وتعتبر نتائجنا امتداداً، تحسيناً وتعميماً لنتائج مماثلة لمؤلفين عديدين.