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A multistage variational iteration method for approximate-analytic solution of avian-human influenza epidemic model

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ABSTRACT

In this paper, the approximate solution of avian-human influenza epidemic model is obtained by a reliable algorithm based on an adaptation of the standard variational iteration method (VIM), which is called the multi-stage variational iteration method (MVIM). A comparison between MVIM and the Runge-Kutta method (RK-method) reveals that the proposed technique is a promising tool to solve the considered problem.

Keywords: Avian-human influenza epidemic model; variational iteration method; numerical solution.

INTRODUCTION

The behavior of solution of avian-human influenza epidemic model is examined in this study (Iwami *et al.* 2007). The components of the basic six-component model are susceptible birds, infected birds with avian influenza, susceptible human, infected human with avian influenza, infected human with mutant avian influenza and recovered human from mutant avian influenza thatare denoted respectively by $X(t)$, $Y(t)$, $S(t)$, $B(t)$, $H(t)$ and $R(t)$ as follows:

$$
\begin{aligned}
\frac{dX}{dt} &= c - bX - \omega XY, \\
\frac{dY}{dt} &= \omega XY - (b+m)Y, \\
\frac{dS}{dt} &= \lambda - \mu S - \beta_1 YS - \beta_2 HS, \\
\frac{dB}{dt} &= \beta_1 SY - (\mu + d + \varepsilon)B, \\
\frac{dH}{dt} &= \beta_2 SH + \varepsilon B - (\mu + \alpha + \gamma)H, \\
\frac{dR}{dt} &= \gamma H - \mu R.
\end{aligned}
\tag{1}
$$

The model (1) has several parameters that must be assigned for numerical simulations. The descriptions and numerical values of the parameters are summarized in Table 1. These descriptions and values were obtained from (lwami *et al*. 2007).

Table 1. The values of the parameters in avian-human influenza epidemic model

Parameter	Value	Description
c	26.5	the rate at which new birds are born
b	5	death rate of susceptible and infected birds
m	5	the additional death rate mediated by avian influenza
w	2	the rate at which avian influenza is contracted from an average bird individual
λ	3	the rate at which new humans are born
μ	0.015	death rate of susceptible and infected humans
d	1	the additional death rate mediated by avian influenza
β_1	0.2	the rate at which avian influenza is contracted from an average bird individual
β_2	0.003	the rate at which mutant avian influenza is contracted from an average human individual
γ	0.01	the recovery rate
ε	0.001	the mutation rate
α	0.06	the additional death rate mediated by mutant avian influenza

This was done with the standard parameter values given above and initial values $X(0)= 10$, $Y(0)= 2$, $S(0)= 100$, $B(0)= 0$, $H(0)= 0$ and $R(0)= 0$. for the six-component model. Our motivation is to concentrate on the application of the multi-stage variational iteration method (MVIM) to a model for avian-human influenza epidemic model given in (1).

It should be noted that one of the main advantages of MVIM is its ability in providing us a continuous representation of the approximate solution, which allows better information of the solution over the time interval. The Runge-Kutta method, on the other hand, provides solutions in discretized form, only at two ends of the time interval, thereby making it complicated in achieving a continuous representation. We aim to compare the effectiveness of MVIM against the well-known fourth-order Runge-Kutta method (RK4). The goal of this paper is to extend the application of the analytic variational iteration method (He, 1998a; He, 1998b; He, 1999; He, 2006) to solve a model for avian-human influenza epidemic model given in (1). Batiha *et al.* (2007a) were the first to implement the multistage VIM for solving a class of nonlinear system of ODEs. In addition, various physical applications using VIM were performed by Batiha *et al* .(2007b) and Batiha *et al.* (2007c).

The paper is organized as follows: A brief review of VIM and MVIM is given in Section 2 and 3, respectively. The application of the proposed numerical scheme to model (1) is illustrated in Section 4. The conclusions are then given in the final Section 5.

VARIATIONAL ITERATION METHOD

According to the variational iteration method (He, 1998a), we consider the following differential equation:

$$Lu + N(u) = g(t), \tag{2}$$

where L is a linear operator, N is a non-linear operator, and $g(t)$ is an inhomogeneous term. Then, we can construct a correct functional as follows:

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_{t_0}^{t} \lambda\left\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\right\}ds, \tag{3}$$

where λ is a general Lagrangian multiplier (He, 1998a), which can be identified optimally via variational theory. The second term on the right is called the correction and $\tilde{u}_{i,n}$ is considered as a restricted variation, i.e., $\delta\tilde{u}_{i,n} = 0$.

MULTISTAGE VARIATIONAL ITERATION METHOD

For large t, VIM is not a good result to approximate solution of some differential equations. To guarantee validity of approximate solution for large t, in the studies in Batiha *et al.* (2007a); Goh *et al.* (2009a,b,c); Goh *et al.* (2010); Goh *et al.* (2008), a new approach called the MVIM is mentioned. According to this approach, the solution from $[t_0, t)$can be reproduced by subdividing this interval into $[t_0, t), [t_1, t_2), ..., [t_{j-1}, t_j = t)$ and a recursive formula of (6) is applied on each subinterval.

$$u_{i,n+1}(t) = u_{i,n}(t) + \int_{t^*}^{t} \lambda\{Lu_{i,n}(s) + N\tilde{u}_{i,n}(s) - g(s)\}ds, \tag{4}$$

Notice that this strategy gives a new construction of the correction functional (6) with a variable t^*as the lower limit of the integration instead of a fixed lower limit of t_0 in (5).The fixed limits are a norm used in the classical VIM which can be seen in He (2007), He & Wu (2007); He (2008); He *et al.* (2010); Herişanu & Marinca (2010); Batiha *et al.* (2007a,b). The initial approximation in each interval is taken from the solution in the previous interval,

$$u_{i,0}(t) = u_i(t^*) = c_i^* \tag{5}$$

where t_i^* is the left-end point of each subinterval and c_i^* is denoted as the initial approximations for $i = 1, 2, ..., m$. By knowing the first initial conditions, one would be able to solve (6) for all unknowns $u_{i,n}(t),\ (i = 1, 2, ..., m;\ n = 0, 1, ...)$. In order to carry out the iteration in every subinterval of equal length $\triangle(t),\ [t_0, t), [t_1, t_2), , [t_{j-1}, t_j = t)$, we need to know the values of the following:

$$u_{i,0}^*(t) = u_i(t^*) = c_i^*,\ i = 1, 2, , m. \tag{6}$$

This information is typically not directly attainable, but through the initial value $t^* = t_0$, we could derive all the initial approximations. This is done by taking the previous initial approximation from the n th-iterate of the preceding subinterval given by (5), i.e.

$$u_{i,0}^*(t) \cong u_{i,n}(t^*),\ i = 1, 2, ..., m \ and\ t^* \in (t_0, t_1). \tag{7}$$

APPLICATION

In this section, we apply the variational iteration method to nonlinear ordinary differential systems (1). According to the variational iteration method, we derive a correct functional as follows:

$$
\begin{aligned}
X_{n+1}(t) &= X_n(t) + \int_{t^*}^{t} \lambda_1 \Big\{ X_n^{'}(\xi) - c + b\tilde{X}_n + \omega \tilde{X}_n \tilde{Y}_n \Big\} d\xi, \\
Y_{n+1}(t) &= Y_n(t) + \int_{t^*}^{t} \lambda_2 \Big\{ Y_n^{'}(\xi) - \omega \tilde{X}_n \tilde{Y}_n + (b+m)\tilde{Y}_n \Big\} d\xi, \\
S_{n+1}(t) &= S_n(t) + \int_{t^*}^{t} \lambda_3 \Big\{ S_n^{'}(\xi) - \lambda + \mu \tilde{S}_n + \beta_1 \tilde{S}_n \tilde{Y}_n + \beta_2 \tilde{S}_n \tilde{H}_n \Big\} d\xi, \\
B_{n+1}(t) &= B_n(t) + \int_{t^*}^{t} \lambda_4 \Big\{ B_n^{'}(\xi) - \beta_1 \tilde{S}_n \tilde{Y}_n + (\mu + d + \varepsilon) \Big\} d\xi, \\
H_{n+1}(t) &= H_n(t) + \int_{t^*}^{t} \lambda_5 \Big\{ H_n^{'}(\xi) - \beta_2 \tilde{S}_n \tilde{H}_n - \varepsilon \tilde{B}_n + (\mu + \alpha + \gamma)\tilde{H}_n \Big\} d\xi, \\
R_{n+1}(t) &= R_n(t) + \int_{t^*}^{t} \lambda_6 \Big\{ R_n^{'}(\xi) - \gamma \tilde{H}_n + \mu \tilde{R}_n \Big\} d\xi.
\end{aligned}
\tag{8}
$$

where λ_1, λ_2, λ_3, λ_4, λ_5 *and* λ_6 are general Lagrange multipliers, $\tilde{X}_n(\xi)$, $\tilde{Y}_n(\xi)$, $\tilde{S}_n(\xi)$, $\tilde{B}_n(\xi)$, $\tilde{H}_n(\xi)$ and $\tilde{R}_n(\xi)$ denote restricted variations, i.e.

$$\delta\tilde{X}_n(\xi) = \delta\tilde{Y}_n(\xi) = \delta\tilde{S}_n(\xi) = \delta\tilde{B}_n(\xi) = \delta\tilde{H}_n(\xi) = \delta\tilde{R}_n(\xi) = 0.$$

Making the above correction functional stationary, we can obtain the following stationary conditions:

$$
\begin{aligned}
&\lambda_1^{'}(\xi) = 0, \\
&1 + \lambda_1(\xi)|_{\xi=t} = 0, \\
&\lambda_1^{'}(\xi) = 0, \\
&1 + \lambda_1(\xi)|_{\xi=t} = 0, \\
&\lambda_2^{'}(\xi) = 0, \\
&1 + \lambda_2(\xi)|_{\xi=t} = 0, \\
&\lambda_3^{'}(\xi) = 0, \\
&1 + \lambda_3(\xi)|_{\xi=t} = 0, \\
&\lambda_4^{'}(\xi) = 0, \\
&1 + \lambda_4(\xi)|_{\xi=t} = 0, \\
&\lambda_5^{'}(\xi) = 0, \\
&1 + \lambda_5(\xi)|_{\xi=t} = 0, \\
&\lambda_6^{'}(\xi) = 0, \\
&1 + \lambda_6(\xi)|_{\xi=t} = 0.
\end{aligned}
\tag{9}
$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = -1. \tag{10}$$

Substituting Eq. (8) into the correction functional Eq. (10) results in the following iteration formula:

$$\begin{aligned}
X_{n+1}(t) &= X_n(t) - \int_{t^*}^{t} \left\{ X_n'(\xi) - c + b\tilde{X}_n + \omega \tilde{X}_n \tilde{Y}_n \right\} d\xi, \\
Y_{n+1}(t) &= Y_n(t) - \int_{t^*}^{t} \left\{ Y_n'(\xi) - \omega \tilde{X}_n \tilde{Y}_n + (b+m)\tilde{Y}_n \right\} d\xi, \\
S_{n+1}(t) &= S_n(t) - \int_{t^*}^{t} \left\{ S_n'(\xi) - \lambda + \mu \tilde{S}_n + \beta_1 \tilde{S}_n \tilde{Y}_n + \beta_2 \tilde{S}_n \tilde{H}_n \right\} d\xi, \\
B_{n+1}(t) &= B_n(t) - \int_{t^*}^{t} \left\{ B_n'(\xi) - \beta_1 \tilde{S}_n \tilde{Y}_n + (\mu + d + \varepsilon) \right\} d\xi, \\
H_{n+1}(t) &= H_n(t) - \int_{t^*}^{t} \left\{ H_n'(\xi) - \beta_2 \tilde{S}_n \tilde{H}_n - \varepsilon \tilde{B}_n + (\mu + \alpha + \gamma)\tilde{H}_n \right\} d\xi, \\
R_{n+1}(t) &= R_n(t) - \int_{t^*}^{t} \left\{ R_n'(\xi) - \gamma \tilde{H}_n + \mu \tilde{R}_n \right\} d\xi.
\end{aligned} \tag{11}$$

The results obtained by 2-term MVIM with time step $\triangle t = 0.1$ and RK4 method with time step $\Delta t = 0.001$ for $X(t),\ Y(t),\ S(t),\ B(t),\ H(t)$ and $R(t)$ are presented in Fig 1 and Table 1.

Figure 1 suggests that the pandemic will occur if the human does not prevent the spread of avian influenza. Fig. 1(a) describes the birds size susceptibility. Fig. 1(b) describes the birds size infected with avianinfluenza. It interprets that the infected birds with avian influenza are endemic just after the incidence of avian influenza in the bird world.

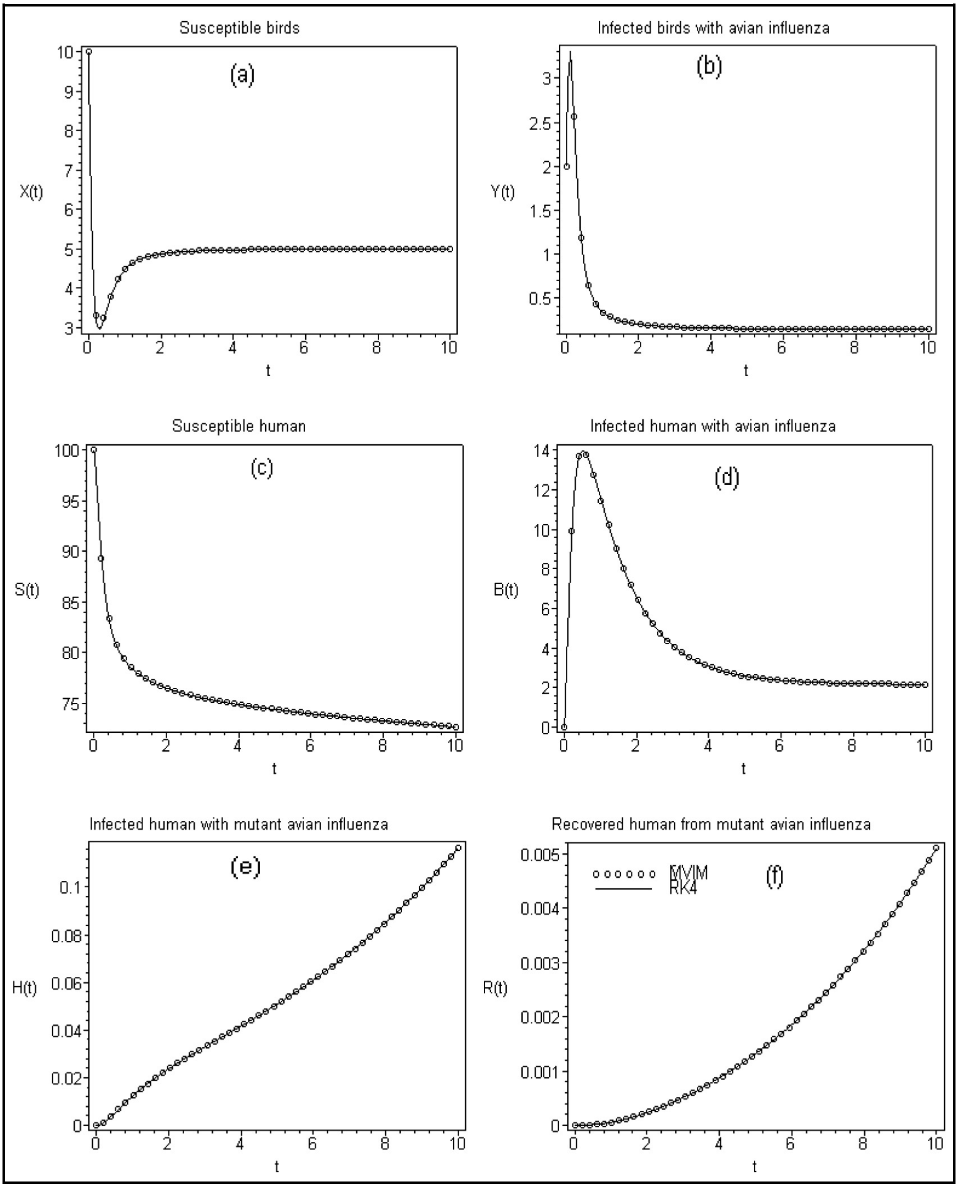

Fig. 1. Graphical comparisons between 2-term MVIM with time step $\triangle t = 0.1$and RK4 method with time step $\Delta t = 0.001$ for $X(t)$, $Y(t)$, $S(t)$, $B(t)$, $H(t)$ and $R(t)$. vs t, respectively.

Figure 1(c) describes the humans size susceptibility, figure 1(d) describes the humans size infected with avian influenza. It interprets that the infected humans with avianinfluenza appear to be pandemic initially and afterward are kept at a low level. Figure 1(e) describes the humans size infected with mutant avian influenza. This indicates that the infected humans with a mutant avian influenza suddenly outbreak and afterward keep the relatively high level of the size. Figure 1(f) describes the recovered humans size infected with mutant avian influenza.Therefore, mathematical model warns that the second outbreak by mutant avian influenza will occur if the human does not prevent the spread ofthe avian influenza.

Table 2. Numerical comparisons between the 2-term MVIM and RK4 solutions.

$\lvert MVIM_{0.1} - RK4_{0.001} \rvert$						
t	$X(t)$	$Y(t)$	$S(t)$	$B(t)$	$H(t)$	$R(t)$
0.0	0.000e-01	0.000e-01	0.000e-01	0.000e-01	0.000e-01	0.000e-01
1.0	5.238e-03	4.018e-04	3.963e-02	1.396e-06	3.997e-02	5.228e-08
2.0	7.413e-04	3.356e-04	2.951e-02	1.970e-05	8.761e-03	6.178e-08
3.0	2.764e-04	1.234e-04	2.497e-02	2.451e-05	7.103e-04	2.858e-07
4.0	1.202e-04	5.497e-05	2.258e-02	2.502e-05	9.877e-04	5.285e-07
5.0	5.777e-05	2.666e-05	2.098e-02	2.433e-05	1.137e-03	7.659e-07
6.0	2.904e-05	1.346e-05	1.977e-02	2.313e-05	9.761e-04	9.903e-07
7.0	1.492e-05	6.931e-06	1.875e-02	2.145e-05	8.068e-04	1.197e-06
8.0	7.754e-06	3.605e-06	1.785e-02	1.914e-05	6.827e-04	1.381e-06
9.0	4.050e-06	1.885e-06	1.702e-02	1.604e-05	5.987e-04	1.536e-06
10.0	2.124e-06	9.881e-07	1.625e-02	1.196e-05	5.416e-04	1.653e-06

From the results given in Table 2, it can be concluded that the rate of convergence and accuracy of MVIM are very good.

CONCLUSION

In this paper, multistage variational iteration method was used for finding the solutions of nonlinear ordinary differential equation systems such as a navian-human influenza epidemic model. We demonstrated the accuracy and efficiency of this method by solving for the considered ordinary differential equation system. The numerical results obtained from the MVIM and the classical fourth-order Runge-Kutta (RK4) method are in complete agreement on this.

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طريقة تكرير تغييرية متعددة المراحل من أجل حل تحليلي تقريبا لنموذج وباء إنفلونزا الطيور / البشر

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خلاصة

في هذا البحث، تم إيجاد حل تقريبي لنموذج وباء إنفلونزا الطيور/ البشر وذلك بوساطة خوارزمية موثوقة تقوم على تكييف لطريقة التكريرالتغييرية القياسية (VIM)، التي تسمى طريقة التكرير التغييرية المتعددة المراحل (MVIM). وبمقارنة الطريقة MVIM وطريقة رونج - كوتا (طريقة - RK) يتبين أن التقنية المقترحة هي أداة واعدة لحل المسألة المطروحة.

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