Kuwait J. Sci. Eng. **37** (1A) pp. 1-14, 2010

Comparative study of multigrid methods and neural network methods for boundary value problems with irregular boundaries

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ABSTRACT

In this paper multigrid methods and neural network methods were used to solve the partial differential equations (PDEs) with boundary conditions (Dirichlet or Neumann) defined on boundaries with simple geometry have been successfully treated on two-dimensional and three-dimensional PDEs. According to the results, multigrid method seems to have certain advantages both in efficiency and in, interpolation accuracy.

Keywords: Boundary value problems, irregular boundaries, neural networks, multi-grid methods, partial differential equations.

1. INTRODUCTION

The goal of our report is to compare between the multigrid method and the neural networks method for solving PDEs. The solution of these problems are required for the resulting set of nonlinear equations. Instead, our report considers the use of the multigrid iterative method. In particular, we employ a nonlinear multigrid method and resent a view of available neural networks and a numerical comparison on two methods.

Neural Networks have been employed before to solve boundary value problems as well as eigenvalue problems (see Lagaris *et al.,* 1998, 2000) They were been used a kind of irregular boundaries (with either Dirichlet or Neumann boundary conditions). Their approach is based on the use of feed forward artificial neural networks (ANNs) whose approximation capabilities have been widely acknowledged (see S. Kevin 2006, K. Jianye *et al.*, 2002 and Lagaris *et al.,* 1998, 2000). More specifically, the proposed approach is based on the synergy of two feed forward ANNs of different types: a multilayer perceptron (MLP) as the basic approximation element and a radial basis function (RBF) network used to satisfy the BCs. Their approach relies on the availability of efficient multidimensional optimization software, (see A. Likas *et al.,* 1998), that is used for the neural network training. We will examine PDEs of the form.

$$L\Psi = f \tag{1}$$

where L is a differential operator and $\Psi = \Psi(x), x \in C$ with Dirichlet or Neumann boundary conditions (BCs). The boundary can be any arbitrarily complex geometrical shape. They considered that the boundary is defined as a set of points that are chosen so as to represent its shape with reasonable accuracy (see A. Likas *et al.*, 1998). The model based on the (MLP)-(RBF) synergy satisfies exactly the BCs but is computationally expensive since at every evaluation of the model one needs to solve a linear system which may be quite large. Moreover, since many efficient optimization methods need the gradient of the objective function, one has to solve an additional linear system of the same order for each gradient component by the penalty method which is very efficient, but does not satisfy exactly the BCs. In practice a combination of these two methods may be, (see Lagaris *et al.,* 1998, 2000), used profitably: the penalty method is used to obtain a reasonable model that satisfies the BCs approximately and is then refined using the synergy method for a few iterations. This is done mainly in order to ensure that the BCs are satisfied exactly. When solving problems requiring several hundreds of boundary points (and thousands of domain points) the method may become relatively slow. They (see Lagaris *et al.*, 1998, 2000) had in their methods, h (the space of grids) having all its components equal to 0.1 (see A. Likas *et al.*, 1998 and S. Kevin McFall, 2006).

1.1 Multigird Methods for Boundary Value Problems

Consider a system described by n unknowns defining the solution of an algebraic system. The best possible (optimal) solver is one that provides this solution requiring a number of operations which is linearly proportional to N (see D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985, and U. Trottenberg, *et al.,* 2001). In a multigrid context this concept is some- what refined by the following two remarks (see D. Jespersen, 1984): The amount of computational work should be proportional to the amount of real physical changes in the computed system; and the solution of many problems is made of several components with different scales, which interact with each other. Therefore, in the discrimination of continuous problem, one should required that the number of unknowns representing the solution should be proportional to the number of the physical features to be described. So, for example, a smooth function can be represented by a few of its values, whereas high resolution is required for highly oscillating functions (see A. Brandt, 1984, W. L. Briggs *et al.,* 2000, W. Hackbusch, 1985, and U. Trottenberg, *et al.,* 2001). As a consequence, to describe a solution of a given problem the use of many scales of discretization is appropriate to represent all components of the solution (see D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985 and U. Trottenberg, *et al.,*

2001). This is one essential aspect of the multigrid strategy. Multigrid methods are known for their efficiency in solving large problems arising from the discretization of PDEs, both linear and nonlinear (see A. Brandt, 1984, W. Hackbusch, 1985, and U. Trottenberg, *et al.,* 2001), on N grid points in O(N) operations. Linear multigrid algorithms have been considered and applied to the PDEs by several authors. Even nonlinear equations can be solved with comparable speed. In all these applications the multigrid algorithm is used for solving the PDEs that aris from the minimum L_2-norm formulation. Eliminating errors in different parts on a sequence of coarse grids, or more generally, coarse subspaces. The basic principle is based on the interplay of smoothing and coarse grid correction which complement each other; the smooth errors not being reduced by smoothing are eliminated by coarse grid corrections. These techniques can generally be applied directly to PDEs but are of most interest when applied to the linear systems arising from their discretizations. Multigrid has proved itself as a powerful and successful numerical technology for fast and efficient computations. In contrast with many other iterative methods, multigrid offers the capability of solving PDE problems with complexity (see D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985, and U. Trottenberge, *et al.,* 2001).

1.2 Full Multigrid Algorithm

So far we have described multigrid as an iterative scheme, where one starts with some initial guess on the finest grid and carries out enough cycles (V-cycles, W-cycles,..) (See D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985, and U.

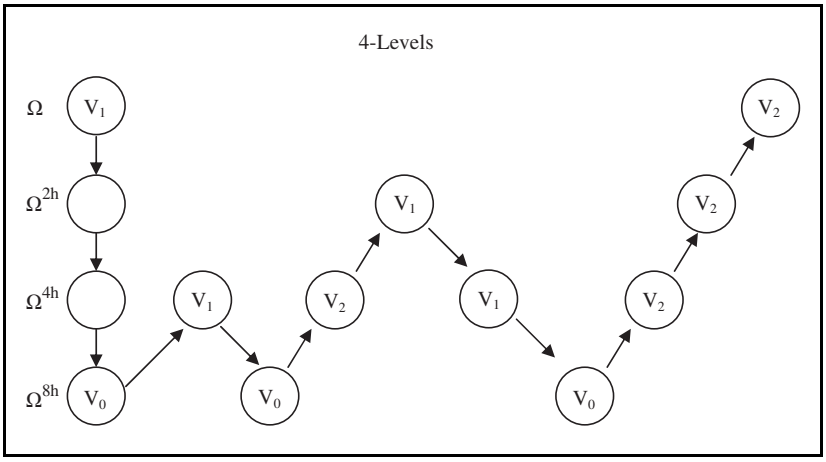

Figure 1: Schematic of the FMG-cycle for 4-level multigrid where ν_0 denotes the number of relaxations on the coarsest mesh (8h), ν_1 No. of relaxation steps before coarse - grid correction, ν_2 No. of relaxation steps after coarse - grid correction

Trottenberg, *et al.,* 2001), to achieve convergence. This is the simplest way to use multigrid: Simply apply enough cycles until some appropriate convergence criterion is met. However, efficiency can be improved by using the Full Multigrid Algorithm (FMG), also known as nested iteration. Instead of starting with an arbitrary approximation on the finest grid (e.g., $\Psi_h = 0$), the first approximation is obtained by interpolating from a coarse-grid solution:

$$\Psi_h = P_H \tag{2}$$

The interpolation operator used in the FMG scheme is called FMG interpolator. Because of the improvement on the initial solution at each starting level, the FMG scheme results to be cheaper than the iterative application of the multigrid cycle without FMG initialization. The coarse-grid solution itself is found by a similar FMG process from even coarser grids. At the coarsest level, we start with the exact solution. Rather than proceed as in Figure (1), then, FMG gets to its solution by a series of increasingly tall "N's", each taller one probing a finer grid figure (1). Note that P in need not be the same P used in the multigrid cycles. It should be at least of the same order as the discretization L_h, but sometimes a higher-order operator leads to greater efficiency. It turns out that we usually need one or at most two multigrid cycles at each level before proceeding down to the next finer grid. While there is theoretical guidance on the required number of cycles (e.g., see W. Hackbusch, 1985, and U. Trottenberg, *et al.*, 2001), we can easily determine it empirically. Fixing the finest level and studying the solution values as we increase the number of cycles per level. The asymptotic value of the solution is the exact solution of the difference equations. The difference between this exact solution and the solution for a small number of cycles is the iteration error. The simple multigrid iteration (cycle) needs the right-hand side f only at the finest level. FMG needs f at all levels.

2. NONLINEAR MULTI-GRID: THE FAS ALGORITHM

The nonlinear PDEs, as

$$L_h(\Psi) = f \tag{3}$$

Consider a symmetric nonlinear PDE, discretized on a grid of mesh-size h:

$$L_h(\Psi_h) = f_h \tag{4}$$

where L_h is the discrete nonlinear matrix operator, f_h is a given vector, and Ψ_h is the vector of unknowns. Let v_h denote the current approximation to Ψ_h, and let

$e_h = \Psi_h - \upsilon_h$ denote the corresponding error. The residual equation is given by

$$r_h = f_h - L_h(\upsilon_h) \tag{5}$$

Substituting $L_h(\Psi_h)$ for f_h in (4) produces

$$L_h(\Psi_h) - L_h(\upsilon_h) = L_h(\upsilon_h + e_h) - L_h(\upsilon_h) = r_h \tag{6}$$

There are two classical multi-grid approaches for solving (4). One method (see D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985, and U. Trottenberg, *et al.*, 2001), which we refer to as Global Linearization (G_L), is to replace the left-hand side by a linear approximation, $L_h\upsilon_h$, obtaining,

$$L_H(\Psi_H) - L_H(\Re\upsilon_h) = \Re e_h. \tag{7}$$

This equation is solved (approximately) by a linear multi-grid solver, and the resulting approximate e_h is added to υ_h. This process, which constitutes a single iteration, is repeated until some convergence criterion is satisfied.

$$L_h(\Psi_h) = f_h \tag{8}$$

We call grid h the "fine grid", and also define on the same domain a "coarse grid" with mesh-size H. The fine-grid equation (6) is first relaxed by some error-smootihing method, such as Gauss-Seidel relazation. Then, a coarse-grid problem is defined:

$$L_H(\Psi_H - \Re_H^h\upsilon_h) = I_H^h r_h \tag{9}$$

are some fine-to-coarse transfer operators (restrictions). After this equation is solved approximately (recursively), the new fine-grid approximation is obtained by interpolating and adding the coarse-grid correction:

$$\nu_h^{new} = \nu_h^{old} + I_h^H(\Psi_H - \tilde{I}_h^H\nu_h) \tag{10}$$

Here, I_h^H is a coarse-to-fine transfer operator (prolongation). Usually, this is followed by additional relaxation on (26). A second approach, (see D. Braess, 1995, A. Brandt, 1984, W. Hackbusch, 1985, and U. Trottenberg, *et al.*, 2001), which we refer to is Local Linearization (LL), is the Full Approximation Scheme (FAS) due to Brandt, 1984. In this method, the error in the fine-grid solution is sommthed using a nonlinear relaxation method, which employs only a local linearization Let L_H denote a nonlinear coarse-grid approximation to L_h, obtained by rediscretizing the PDE on the coarse grid. The coarse-grid problem is given by

$$\tilde{L}_H\Psi_H = \tilde{L}_H - \frac{\tilde{I}_h^H(\nu_h)}{I_h^H r_h} \tag{11}$$

The correction to the fine-grid approximation is added as in (5). One way of

solving nonlinear problems with multi-grid is to use Newton's method, which produces linear equations for the correction term at each iteration. We can then use linear multi-grid to solve these equations. A great strength of the multi-grid idea, however, is that it can be applied directly to nonlinear problems. All we need is a suitable nonlinear relaxation method to smoooth the errors, plus a procedure for approximating corrections on coarser grids. This direct approach is Brandt's Full Approximation Storage Algorithm (FAS). No nonlinear equations need be solved, except perhaps on the coarsest grid. To develop the nonlinear algorithm, suppose we have a relaxation procedure that can smooth the residual vector as in the linear case. To find υ_h that

$$L_h(\Psi_h + \nu_h) = f_h \tag{12}$$

3. EXAMPLES

3.1 Tow Dimensional Problems

Problem 1: Consider the linear problem $\nabla^2\Psi(x, y) = \exp^{-x}(x - 2 + y^3 + 6y)$, $x, y \in [0, 1]$. the analytic solution is: $\Psi_a(x, y) = \exp^{-x}(x + y^3)$. This example has been treated in (see Lagaris *et al.*, 1998, 2000) by a simpler neural network model by picking points on the square boundary as if it were an irregular shape. More specifically, They consider points $(x; y)$ on the boundary, by dividing the interval $[0; 1]$ on the x-axis and y-axis repectively using equidistant points. The total number of points taken on the boundary is M = 36. Inside the definition domain they pick points on a rectangular grid by subdividing the $[0; 1]$ interval in 10 equal subintervals that correspond to 9 points in each direction. Thus a total of K = 81 points are selected. And they using 20 hidden units.

Table 1: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (1) with the Dirichlet boundary condition, and we use the linear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.741*D* (-03)	0.621 *D* (-03)	0.125 *D* (-03)	0.331 *D* (-03)
1/8	0.151 *D* (-04)	0.234 *D* (-04)	0.431 *D* (-04)	0.513 *D* (-04)
1/16	0.214 *D* (-05)	0.181 *D* (-05)	0.217 *D* (-05)	0.232 *D* (-05)
1/32	0.264 *D* (-06)	0.152 *D* (-06)	0.523 *D* (-06)	0.636 *D* (-06)
1/64	0.113 *D* (-07)	0.113 *D* (-07)	0.261 *D* (-07)	0.364 *D* (-07)

Problem 2: $\nabla^2\Psi(x,y) + \Psi(x,y) = (1 + x^2 + y^2) + \frac{4}{(1+x^2+y^2)^2}$. with the analytical solution $\Psi_a(x,y) = \log(1 + x^2 + y^2)$, with the cardioid domain they used M = 100 boundary points and K = 500 grid points displayed. An MLP with 20 hidden units was used. The accuracy of the obtained solution (with Neumann BCs) at a dense grid of interpolation points is, the results are similar for the case of Dirichlet BCs. They found that the accuracy of the solution at these intermediate test points is maintained at the level of the neighboring training ones.

Problem 3: $\nabla^2\Psi(x,y) + \Psi(x,y)\partial\Psi(x,y)/\partial y = \sin(\pi x)(2 - \pi^2 y^2 + 2y^3\sin(\pi x))$. with the analytical solution $\Psi_a(x,y) = \log(1 + x^2 + y^2)$. with $(x,y) \in [0,1]$, and with mixed BCs: $\Psi(0,y) = 0, \Psi(1,y) = 0, \Psi(x,0) = 0, \partial\Psi(x,1)/\partial y = 2(\pi x)$. The exact solution is $\Psi_a(x,y) = y^2\sin(\pi x)$.

3.1.1 The multigrid Technique For The Problem in 2D

Because the multigrid method needs a sequence of grids, then replacing each term in (1) by a second order finite difference approximation. The discretized equation is: $\Psi_{i+1,j} + \Psi_{i-1,j} + \Psi_{i,j+1} + \Psi_{i,j-1} - 4\Psi_{i,j} = h^2\exp^{-i}(i - 2 + j^3 + 6j)$. Also we used a full multigrid method $\gamma(\nu_1,\nu_2)$, we mean by ν_1,ν_2 the number of relaxation steps before and after the coarse-grid correction reprctively, take $\nu_1 = \nu_2 = 2$. For this linear problem we used a linear interpolation and a half weight restriction operator as the form: $(\Re\Psi)_{i,j} = \frac{1}{2}(\Psi_{2i,2j}) + \frac{1}{4}(\Psi_{2i+1,2j} + \Psi_{2i-1,2j} + \Psi_{2i-1,2j} + \Psi_{2i,2i+1} + \Psi_{2i,2j-1})$. The results of the mgm for problem (1) are reported in the talbe (1).

Table 2: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (2) with the Dirichlet boundary condition, and we use the nonlinear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing, and we use Newton interpolation

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.511*D* (-02)	0.821 *D* (-02)	0.221 *D* (-02)	0.121 *D* (-02)
1/8	0.332 *D* (-03)	0.451 *D* (-03)	0.351 *D* (-03)	0.812 *D* (-03)
1/16	0.641 *D* (-04)	0.413 *D* (-04)	0.911 *D* (-04)	0.413 *D* (-04)
1/32	0.619 *D* (-05)	0.715 *D* (-05)	0.217 *D* (-05)	0.516 *D* (-05)
1/64	0.413 *D* (-06)	0.516 *D* (-06)	0.421 *D* (-06)	0.321 *D* (-06)

Table 3: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (2) with the Neumann boundary condition, and we use the nonlinear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing, and we use Newton interpolation

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.511*D* (-02)	0.821 *D* (-02)	0.221 *D* (-01)	0.121 *D* (-01)
1/8	0.332 *D* (-03)	0.451 *D* (-03)	0.351 *D* (-02)	0.812 *D* (-02)
1/16	0.641 *D* (-04)	0.413 *D* (-04)	0.911 *D* (-03)	0.413 *D* (-03)
1/32	0.619 *D* (-05)	0.715 *D* (-05)	0.217 *D* (-04)	0.516 *D* (-04)
1/64	0.413 *D* (-06)	0.516 *D* (-06)	0.421 *D* (-05)	0.321 *D* (-05)

Table 4: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (3) with the Dirichlet boundary condition, and we use the nonlinear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing, and we use Newton interpolation

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.511*D* (-02)	0.821 *D* (-02)	0.221 *D* (-02)	0.121 *D* (-02)
1/8	0.332 *D* (-03)	0.451 *D* (-03)	0.351 *D* (-03)	0.812 *D* (-03)
1/16	0.641 *D* (-04)	0.413 *D* (-04)	0.911 *D* (-04)	0.413 *D* (-04)
1/32	0.619 *D* (-05)	0.715 *D* (-05)	0.217 *D* (-05)	0.516 *D* (-05)
1/64	0.413 *D* (-06)	0.516 *D* (-06)	0.421 *D* (-06)	0.321 *D* (-06)

3.1.2 The non-linear multigrid Technique For Problem 2

We apply the orthogonal transformation

$$\Psi = \begin{pmatrix} x(r,\phi) = r\cos\phi \\ y(r,\phi) = r\sin\phi \end{pmatrix}$$

The transformed Laplacian in polar coordinates as: $\Psi_{rr} + 1/r\Psi_r + 1/r^2\Psi_{\phi\phi} + \Psi(r,\phi) = f(r,\phi)$. The discretization of the nonlinear equation (problem 2) as the

form: $[1+\frac{1}{2}\sin 2\phi+\frac{1}{2i}(1-\frac{1}{2}\sin 2\phi)]\Psi_{i+1,j}+[1+\frac{1}{2}\sin 2\phi-\frac{1}{2i}(1-\frac{1}{2}\sin 2\phi)]\Psi_{i-1,j}$ $+\left[\frac{1}{\partial\phi^2}(1-\frac{1}{2}\sin 2\phi-\frac{\cos 2\phi}{2i^2\partial\phi})\right]\Psi_{i,j+1}+\left[\frac{1}{\partial\phi^2}(1-\frac{1}{2}\sin 2\phi-\frac{\cos 2\phi}{2i^2\partial\phi}\right]\Psi_{i,j-1}-2[1+1/2(\sin 2\phi)$ $+1/2(1-\frac{1}{2}\sin 2\phi)+h^2]\Psi_{i,j}=h^2 f_{ij}$. Where f_{ij}, is the right hand side multiplying by h^2. The nonlinear Gauss-Seidel schemes solves the above discretizied equation for $\psi_{i,j}^{new}$. We replace the equation by one step of a Newton iteration: $\Psi_{i,j}^{new}=\Psi_{i,j}^{old}-\frac{L(\Psi_{i,j}^{old})-f_{i,j}}{\partial L\Psi_{i,j}^{old}/\partial\Psi_{i,j}}$., where $f_{i,j}$ is right hand side multiplying by h^2. We have used a full weight restriction operator for the residual correction. The accuracy of the mgm for problem (2) is replaced in table (2)

Table 5: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (3) with the Neumann boundary condition, and we use the nonlinear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing, and we use Newton interpolation

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.413*D* (-02)	0.711 *D* (-02)	0.152 *D* (-01)	0.314 *D* (-01)
1/8	0.141 *D* (-03)	0.324 *D* (-03)	0.421 *D* (-02)	0.163 *D* (-02)
1/16	0.516 *D* (-04)	0.348 *D* (-04)	0.381 *D* (-03)	0.813 *D* (-03)
1/32	0.716 *D* (-05)	0.825 *D* (-05)	0.172 *D* (-04)	0.621 *D* (-04)

3.2 Three Dimensional Problem

Problem 4: We considered the problem: $\nabla^2\Psi(x,y,z)=\Psi^2(x,y,z)+\exp(x)y^2+$ $z^2\sin(y)-(\exp(x)y^2+(z^2-2)\sin(y))^2$. The domain, [16], is described in spherical coordinates (r,ϕ,θ) as: r in $[0.5,1]$, ϕ in $[0,\pi/2]$, θ in $[0,\pi/2]$. The problem had solved using Cartesian coordinates $(x;y;z)$[16]. The analytical solution is $\Psi_\alpha(x,y,z)=\exp(x)y^2+(z^2-2)\sin(y)$ they considered M = 176 boundary points and K = 729 grid points and solve the nonlinear equation with both Dirichlet and Neumann BCs. The obtained solutions using an MLP with 40 hidden units are accurate with absolute error value less than 10^{-5} (see Lagaris *et al.*, 1998, 2000).

3.2.1 The multigrid Technique For The Problem in 3D

We apply the orthogonal transformation

$$\Psi = \begin{pmatrix} x(r,\phi,\theta) = r\ \sin\phi\ \cos\theta \\ y(r,\phi,\theta) = r\ \sin\phi\ \sin\theta \\ z(r,\phi,\theta) = r\ \cos\phi \end{pmatrix}$$

to map the region $\Omega^* = \{(r,\phi,\theta) : 0 \le r \le 1, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$. onto unite sphere. The transformed Laplacian is written in spherical coordinates as: $\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\frac{\partial}{\partial\phi}) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}$. Because of multigrid method, one needs a sequences of grids, then replacing each term in the above Laplacian, by a second-order finite difference approximation, we have the discretization equation as the form: $1^2[\Psi_{i+1,j,k} + \Psi_{i-1,j,k} - 2\Psi_{i,j,k}] + \frac{1}{lh^2}[\Psi_{i+1,j,k} - \Psi_{i-1,j,k}] + \frac{1}{(lh\,\partial\phi)^2}[\Psi_{i,j+1,k} + \Psi_{i,j-1,k} - 2\Psi_{i,j,k}] + \frac{\cos\phi}{2(lh^2)\partial\phi\sin\phi}[\Psi_{i,j+1,k} - \Psi_{i,j-1,k}] + \frac{1}{(lh\,\partial\phi\sin\phi)^2}$ $[\Psi_{i,j,k+1} + \Psi_{i,j,k-1} - 2\Psi_{i,j,k}] + \Psi^2_{i,j,k} = f_{i,j,k}$. Where $f_{i,j,k} = f(r,\phi,\theta), r = lh, l = 1,2,3,..,N$. Solving for $\Psi_{i,j,k}$, we get the solution at the N^3 interior points. We also used a full multigrid (non-linear multigrid) method $\gamma(2,2)$, and we also used a Newton interpolation, and a full weight restriction operator for the correction parte. The results are reported in table (5) and (6) for the Drichlet and Neumann boudary value problem.

Table 6: L_2-Norm of Error, Maximum Norm of Error, L_2-Norm of Residual, and Maximum Norm of Residual for problem (4) with the Dirichlet boundary condition, and we use the nonlinear multi-grid method with full multi-grid cycle and point wise Gauss-Siedel scheme for the smoothing, and we use Newton interpolation

h	L_2-norm of E	Max.norm of E	L_2-norm of R	Max.norm of R
1/2	0.000	0.000	0.000	0.000
1/4	0.132 *D* (-02)	0.263 *D* (-02)	0.538 *D* (-02)	0.616 *D* (-02)
1/8	0.711 *D* (-03)	0.119 *D* (-03)	0.817 *D* (-03)	0.313 *D* (-03)
1/16	0.163 *D* (-04)	0.114 *D* (-04)	0.166 *D* (-04)	0.517 *D* (-04)
1/32	0.253 *D* (-05)	0.235 *D* (-05)	0.243 *D* (-05)	0.137 *D* (-05)

4. COMPARISON WITH NEURAL NETWORK

The above problems were also solved with the neural network method. The neural approach assumes a small number of parameters (40 for PDEs), but requires more sophisticated minimization algorithms. In the mgm case, interpolation is performed using a rectangular grid of 23 × 23 equidistant points (test points). It is clear that the solution is not expressed in closed analytical form as in the neural case, but additional computations are required in order to find the value of the solution at an arbitrary point in the domain. Table 7 reports the maximum error corresponding to the neural and to the mgm method at the training and at the interpolation set of points. It is obvious that at the training points the solution of the mgm method is very satisfactory and in some cases it is better than that obtained using the neural method. It is also clear that the accuracy at the interpolation points is orders of magnitude lower as compared to that at the training points. On the contrary, the mgm method provides solutions of excellent interpolation accuracy, since, as Table 7 indicates. It must also be stressed that the accuracy of the mgm method decreases as the size of the grid becomes smaller, and that the neural approach considers a mesh of 10 × 10 points (see B. M. Wilamowski *et al.*, 2001, A. Likas *et al.*, 1998 and S. Kevin McFall, 2006) while, in the mgm case a 18 × 18 mesh was employed. In order to investigate the convergence properties of the method, they conducted several numerical experiments using the non-linear example of problem 2 with Dirichlet BCs. Specifically they calculated the approximation error in the max norm for several choices of the number of the hidden MLP units.

Table 7: Maximum error of the problems for the neural and the multigrid methods as reported in (see A. Likas *et al.*, 2000 and S. Kevin McFall, 2006).

Problem No.	NNs Train.P.	NNs Int.P.	MGM Train.P.	MGM Int.P.
P1	5×10^{-7}	5×10^{-7}	3×10^{-8}	2×10^{-7}
P2	6×10^{-6}	6×10^{-6}	2×10^{-7}	3×10^{-7}
P3	1.5×10^{-5}	1.5×10^{-5}	2×10^{-7}	3×10^{-6}
P4	5×10^{-5}	5×10^{-5}	2×10^{-5}	3×10^{-5}

5. CONCLUSION

We presented a method capable of solving boundary value problems of the Dirichlet and Neumann types, for boundaries that due to their geometrical complexity can only be described via a set of participating points. The accuracy of the obtained solution by the multigrid methods comparing with their solution (see A. Likas *et al.*, 1998, 2000 and S. Kevin McFall, 2006), is very efficiently

and not expensive. The proposed method is quite general and can be used for a wide class of linear and non-linear PDEs. Future work will focus on the application of the method to specific problems, containing real objects with arbitrarily complex boundaries. Interesting problems of this kind arise in many scientific fields. We can conclude that the nonlinear multigrid methods are shown to be very promising for two and three-dimensional PDEs. The mgm is expected to eliminate a large part of the discrepancy between the number of fine-grid cycles required for the nonlinear problem, compared to the linear one. We find that our method very effective and most significantly.

6. ACKNOWLEDGEMENT

The author would like to express her deep thanks to the referees for their interesting suggestions and comments on this paper.

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Submitted : *24/6/2008*
Revised : *24/3/2009*
Accepted : *1/4/2009*

دراسة مقارنة لطرق الشبكات المتعددة والشبكات العصبية لمسائل القيم الحدية ذات الحدود الشاذة

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خلاصة

في هذا البحث، طرق الشبكات المتعددة والشبكات العصبية استخدمت في حل المعادلات التفاضلية الجزئية بشروط دريشلت ونيومان المعرفة على الحدود الهندسية البسيطة، وقد تم علاجها بنجاح للمعادلات التفاضلية الجزئية في بعدين وثلاثة أبعاد. ووفقا للنتائج، فإن طريقة الشبكات المتعددة ذات مميزات معينة في كل من الكفاءة، والاستيفاء والدقة وتلك من خلال الأمثلة العددية التي تم توضيحها في آخر البحث.