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On a new weaker form of Popa's rare continuity via λ-open sets

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ABSTRACT

In 1979, Popa introduced the notion of rare continuity as a generalization of weak continuity. The objective of this paper is to introduce a new generalization of rare continuity called rare λ-continuity and investigate some of its fundamental properties.

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INTRODUCTION

Popa (1979) introduced the interesting notion of rare continuity as a generalization of Levine's weak continuity (1961), and also studied some of its properties. Long and Herrington (1982), Jafari (1995) and Jafari (1997) further investigated rare continuity and some related notions and obtained some new results. The purpose of the present paper is to introduce the concept of rare λ-continuity in topological spaces via λ-open sets as a generalization of rare continuity. We investigate several basic properties of rarely λ-continuous functions. The notion of I.λ-continuity is also introduced which is weaker than λ-continuity and stronger than rare λ-continuity.

Throughout the paper a space will always mean a topological space on which no separation axioms are assumed unless explicitly stated.

Definition 1. A subset A of a space (X,τ) is called

(1) regular open if $A = Int(Cl(A))$. (Stone 1937)

(2) a Λ-set if it is equal to its kernel (= saturated set), i.e. to the intersection of all open supersets of A. (Maki 1986)

(3) λ-closed if $A = U \cap V$, where U is a Λ-set and V is a closed set. (Arenas *et al.* 1997)

(4) λ-open if $X \backslash A$ is λ-closed. (Caldas, Jafari & Navalagi 2007)

The family of all λ-open sets (respectively containing x∈X) will be denoted by λO(X) (respectively λO(X,x)). Recall that a rare set is a set R such that Int(R) = ∅. A point x∈X is said to be a λ-interior point of a subset A⊂X if there exists a λ-open set B containing x such that B⊂A. (Caldas & Jafari 2005)

The set of all λ-interior points of A is said to be λ-interior of A and is denoted by $Int_\lambda(A)$.

Definition 2. A function f:X→Y is called

(1) weakly continuous if for each x∈X and each open set V containing f(x), there exists an open set U containing x such that f(U)⊂Cl(V). (Levine 1961)

(2) rarely continuous if for each x∈X and each open set V containing f(x), there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and an open set U containing x such that $f(U) \subset V \cup R_V$. (Popa 1979)

(3) λ-continuous if the inverse image of every open set in Y is λ-open in X. (Arenas *et al.* 1997, Caldas, Jafari & Navalagi 2007)

A space X is called locally indiscrete if every open set is closed. (Nieminen 1977)

The graph of a function f:X→Y, denoted by G(f), is the subset {(x,f(x)):x∈X} of the product space X×Y.

RARELY λ-CONTINUOUS FUNCTIONS

Definition 3. A function f:X→Y is called rarely λ-continuous if for each x∈X and each open set V containing f(x), there exist a rare set R_V with $V \cap Cl(R_V) = \emptyset$ and U∈ λO(X,x) such that $f(U) \subset V \cup R_V$.

Remark 2.1. The following hold for a function f:X→Y:

weakly continuous ⇒ rarely continuous ⇒ rarely λ-continuous

None of these implications is reversible as shown in the following example and in Popa (1979).

Example 2.2. Let X = Y = {a,b,c}, τ = {X,∅,{a},{a,b}} and σ = {X,∅,{a},{b,c}}. Then the function f:(X,τ) →(Y,σ) defined by f(a) = c, f(b) = c, f(c) = a is rarely λ-continuous but it is not rarely continuous.

Theorem 2.3. For a function f:X→Y, the following are equivalent:

(1) The function f is rarely λ-continuous at x∈X,

(2) For each open set V containing f(x), there exists U∈ λO(X,x) such that Int(f(U)∩(Y\V)) = ∅,

(3) For each open set V containing f(x), there exists U$\in$ λO(X,x) such that Int(f(U))$\subset$Cl(V).

Proof. (1)$\Rightarrow$(2): Let V be an open set containing f(x). Since f(x)$\in$V$\subset$Int(Cl(V)) and Int(Cl(V)) is an open set containing f(x), there exist a rare set R_V with Int(Cl(V))$\cap$Cl(R_V) $= \emptyset$ and a λ-open set U containing x such that f(U)$\subset$Int(Cl(V))$\cup R_V$. This implies that

$$\mathrm{Int}(f(U)\cap(Y\setminus V)) = \mathrm{Int}(f(U))\cap \mathrm{Int}(Y\setminus V)\subset$$

$$\mathrm{Int}(\mathrm{Cl}(V)\cup R_V)\cap(Y\setminus \mathrm{Cl}(V))\subset(\mathrm{Cl}(V)\cup \mathrm{Int}(R_V))\cap(Y\setminus \mathrm{Cl}(V)) = \emptyset.$$

(2)$\Rightarrow$(3): It is obvious.

(3)$\Rightarrow$(1): Let V be an open set containing f(x). Then there exists U$\in$ λO(X,x) such that Int(f(U))$\subset$Cl(V). This implies that

$$f(U) = (f(U)\setminus \mathrm{Int}(f(U)))\cup \mathrm{Int}(f(U))\subset$$

$$(f(U)\setminus \mathrm{Int}(f(U)))\cup \mathrm{Cl}(V) = (f(U)\setminus \mathrm{Int}(f(U)))\cup V\cup(\mathrm{Cl}(V)\setminus V) =$$

$$[(f(U)\setminus \mathrm{Int}(f(U)))\cap(Y\setminus V)]\cup V\cup(\mathrm{Cl}(V)\setminus V).$$

Take $R_1 = (f(U)\setminus \mathrm{Int}(f(U)))\cap(Y\setminus V)$ and $R_2 = \mathrm{Cl}(V)\setminus V$. Then R_1 and R_2 are rare sets and $R_V = R_1\cup R_2$ is a rare set such that Cl(R_V)$\cap$V $= \emptyset$ and f(U)$\subset$V$\cup R_V$. Thus, f is rarely-λ-continuous.

Now we have the following two results whose proofs are omitted since they are immediate consequences of Definition 3.

Theorem 2.4. A function f:X$\rightarrow$Y is rarely λ-continuous at x$\in$X if and only if for each open set V of Y containing f(x), there exists a rare set R_V with Cl(R_V)$\cap$V $= \emptyset$ such that x$\in$$\mathrm{Int}_\lambda(f^{-1}(V\cup R_V))$.

Theorem 2.5. A function f:X$\rightarrow$Y is rarely λ-continuous if and only if for each open set Vof Y, there exists a rare set R_V with Cl(R_V)$\cap$V $= \emptyset$ such that $f^{-1}(V)\subset\mathrm{Int}_\lambda(f^{-1}(V\cup R_V))$.

Definition 4. A function f:X$\rightarrow$Y is said to be strongly λ-open if the image of every λ-open set is open.

Theorem 2.6. If f:X$\rightarrow$Y is a strongly λ-open rarely λ-continuous function and Y is Hausdorff, then f has λ-closed point inverses.

Proof. Let y$\in$Y and x$\in$$\{x\in X:f(x)\neq y\}$. Since f(x)$\neq$y and Y is Hausdorff, there exist disjoint open sets V_1, V_2 such that f(x)$\in V_1$ and y$\in V_2$. Since $V_1\cap V_2 = \emptyset$, then Cl(V_1)$\cap V_2 = \emptyset$. Also, we have y$\notin$Cl(V_1). Since f is strongly λ-open and rarely λ-continuous, there exists a λ-open set U containing x such that Int(f(U)) = f(U)$\subset$Cl(V_1). Suppose that U is not contained in $\{x\in X:f(x)\neq y\}$.

There exists a point $u \in U$ such that $f(u) = y$. Since $f(U) \subset Cl(V_1)$, then $y = f(u) \in Cl(V_1)$. This is a contradiction. Thus, $U \subset \{x \in X : f(x) \neq y\}$ and $\{x \in X : f(x) \neq y\}$ is λ-open in X. Therefore, $\{x \in X : f(x) \neq y\}$ is λ-open in X, equivalently $f^{-1}(y) = \{x \in X : f(x) = y\}$ is λ-closed in X.

Theorem 2.7. If $f : X \to Y$ is rarely λ-continuous and Y is Hausdorff, then for each $(x,y) \notin G(f)$, there exist a λ-open set $V \subset X$ and an open set $U \subset Y$ containing x and y, respectively, such that $Int(f(V)) \cap Int(Cl(U)) = \emptyset$.

Proof. Let $(x,y) \notin G(f)$. We have $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets U and G containing y and f(x), respectively. Then $Int(Cl(U)) \cap Cl(G) = \emptyset$. Since f is rarely λ-continuous, then there exists a λ-open set V containing x such that $Int(f(V)) \subset Cl(G)$. Thus, $Int(f(V)) \cap Int(Cl(U)) = \emptyset$.

Lemma 2.8. A space X is locally indiscrete if and only if every λ-open set of X is open in X. (Caldas *et al*. 2006)

Theorem 2.9. The following are equivalent for a function $f : X \to Y$, where X is a locally indiscrete space:

(1) f is rarely λ-continuous,

(2) f is rarely continuous.

Proof. Obvious.

Definition 5. A function $f : X \to Y$ is called

(1) I.λ-continuous at $x \in X$ if for each open set V containing f(x), there exists $U \in \lambda O(X,x)$ such that $Int(f(U)) \subset V$.

(2) I.λ-continuous on X if f has this property at each point $x \in X$.

Remark 2.10. The following hold for a function $f : X \to Y$:

$$\lambda\text{-continuous} \Rightarrow \text{I.}\lambda\text{-continuous} \Rightarrow \text{rarely } \lambda\text{-continuous}$$

None of these implications is reversible as shown in the following example.

Example 2.11. Let $X = \{a,b,c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then the function $f : (X,\tau) \to (X,\tau)$ defined by $f(a) = a$, $f(b) = a$, $f(c) = c$ is I.λ-continuous but it is not λ-continuous. The function $g : (X,\tau) \to (X,\tau)$ defined by $g(a) = b$, $g(b) = a$, $g(c) = c$ is rarely λ-continuous but it is not I.λ-continuous.

Theorem 2.12. The following are equivalent for a function $f : X \to Y$, where Y is a regular space:

(1) f is I.λ-continuous,

(2) f is rarely λ-continuous.

Proof. (1)⇒(2): Obvious.

(2)⇒(1): Let f be rarely λ-continuous and x∈X. Let V⊂Y be an open set and f(x)∈V. Since Y is regular, then there exists an open set V_1 containing f(x) such that Cl(V_1) ⊂V. Since f is rarely λ-continuous, then by Theorem 2.3 there exists U∈ λO(X,x) such that Int(f(U))⊂Cl(V_1). Thus, Int(f(U))⊂V and hence f is I.λ-continuous.

Definition 6. (Ekici *et al.* 2008) A function f:X→Y is said to be

(1) (λ,s)-open if f(A) is semiopen for every λ-open subset A⊂X.

(2) weakly-λ-continuous if for each x∈X and each open set V containing f(x), there exists U∈ λO(X,x) such that f(U)⊂Cl(V).

Theorem 2.13. If a function f:X→Y is rarely λ-continuous and (λ,s)-open, then f is weakly λ-continuous.

Proof. Let x∈X and V be an open subset of Y containing f(x). Since f is rarely λ-continuous, there exists a λ-open set U of X containing x such that Int(f(U))⊂Cl(V). Since f is (λ,s)-open, then f(U) is semiopen in Y. We have f(U)⊂Cl(Int(f(U)))⊂Cl(V). Hence, f is weakly λ-continuous.

Corollary 2.14. If f:X→Y is a strongly λ-open rarely λ-continuous function, then f is weakly λ-continuous.

Proof. By Theorem 2.13, it follows from the fact that every strongly λ-open function is (λ,s)-open.

Theorem 2.15. If f:X→Y is rarely λ-continuous function and λO(X) is closed under finite intersections, then the graph function g:X→X×Y, defined by g(x) = (x,f(x)) for every x in X is rarely λ-continuous.

Proof. Let x∈X and A be an open set containing g(x). Then there exist open sets U and V in X and Y, respectively, such that (x,f(x))∈U×V⊂A. Since f is rarely λ-continuous, there exists H∈ λO(X,x) such that Int(f(H))⊂Cl(V). Take K = U∩H. Then K is a λ-open set containing x and hence Int(g(K))⊂Int(U×f(H))⊂U×Cl(V)⊂Cl(A). Thus, g is rarely λ-continuous.

Lemma 2.16. If g:Y→Z is continuous and one-to-one, then g preserves rare sets. (Long & Herrington 1982)

Theorem 2.17. If f:X→Y is rarely λ-continuous and g:Y→Z is continuous and one-to-one, then gof:X→Z is rarely λ-continuous.

Proof. Let x∈X and V⊂Z be an open set such that (gof)(x)∈V. Since g is continuous, then there exists an open set H containing f(x) such that g(H)⊂V. Since f is rarely λ-continuous, there exist a rare set R_H with H∩Cl(R_H) = ∅ and a λ-open set U containing x, such that f(U)$\subset$H$\cup$R$_H$. By Lemma 2.16, g(R$_H$) is a rare set in Z. Since R$_H$ $\subset$Y\H and g is injective, then Cl(g(R$_H$))$\cap$V = $\emptyset$. Thus (gof)(U)$\subset$V$\cup$g(R$_H$) and hence gof is rarely λ-continuous.

Definition 7. A function f:X$\rightarrow$Y is called pre-λ-open if f(A) is λ-open in Y for every λ-open set A$\subset$X. (Jafari *et al.* 2008)

Theorem 2.18. If f:X$\rightarrow$Y is pre-λ-open surjection and g:Y$\rightarrow$Z is a function such that gof:X$\rightarrow$Z is rarely λ-continuous, then g is rarely λ-continuous.

Proof. Let y$\in$Y and x$\in$X with f(x) = y. Suppose that V$\subset$Z is an open set containing (gof)(x). Since gof is rarely λ-continuous, then there exist a rare set R$_V$ with V$\cap$Cl(R$_V$) = $\emptyset$ and U$\in$ λO(X,x) such that (gof)(U)$\subset$V$\cup$R$_V$. Take H = f(U). Then H is λ-open containing f(x). This implies that there exist a rare set R$_V$ with V$\cap$Cl(R$_V$) = $\emptyset$ and H$\in$ λO(Y,y) such that g(H)$\subset$V$\cup$R$_V$. Hence, g is rarely λ-continuous.

Definition 8. A space X is called λ-T$_2$ if for x, y$\in$X such that x$\neq$y there exist disjoint λ-open sets U and V such that x$\in$U and y$\in$V. (Caldas, Jafari & Navalagi 2007)

It is shown that λ-T$_2$ is equivalent with T$_0$ in Ganster *et al.* (2008a),

Lemma 2.19. If for each pair of distinct points x$_1$ and x$_2$ in a space X, there exist a function f of X into (Y,σ) such that Y is Urysohn, f(x$_1$) $\neq$f(x$_2$) and f is weakly λ-continuous at x$_1$ and x$_2$, then X is λ-T$_2$. (Ekici *et al.* 2008)

Corollary 2.20. If for each pair of distinct points x$_1$ and x$_2$ in a space X, there exist a function f of X into (Y,σ) such that Y is Urysohn, f(x$_1$) $\neq$f(x$_2$) and f is (λ,s)-open rarely λ-continuous, then X is λ-T$_2$.

Proof. Follows from Theorem 2.13 and Lemma 2.19.

Definition 9. Let A = {V$_i$} be a class of subsets of X. By rarely union sets (Jafari 1995) of A we mean {V$_i$$\cup$R$_{Vi}$}, where each R$_{Vi}$ is a rare set such that each of {V$_i$$\cap$Cl(R$_{Vi}$)} is empty.

Definition 10. A subset B of X is said to be rarely almost compact relative to X (Jafari 1995) if every open cover of B by open sets of X, there exists a finite subfamily whose rarely union sets cover B.

Definition 11.

(1) A topological space X is said to be rarely almost compact if the set X is rarely almost compact relative to X. (Jafari 1995)

(2) A subset A of X is said to be λ-compact relative to X if every cover of A by λ-open sets of X has a finite subcover. (Caldas, Jafari & Navalagi 2007)

Remark 2.21. It should be mentioned that the notion of λ-compactness is called λO-compactness in (Ganster et al. 2008b) and its properties are studied there.

Theorem 2.22. If f:X→Y is rarely λ-continuous and A is a λ-compact set relative to X, then f(A) is rarely almost compact subset relative to Y.

Proof. Let ψ be an open cover of f(A) and let Φ be the family of each set V in ψ such that $V \cap f(A) \neq \emptyset$. This implies that Φ is an open cover of f(A) and hence for each $x \in A$, there exists a $V_x \in \Phi$ such that $f(x) \in V_x$. Since f is rarely λ-continuous, then there exist a rare set R_{Vx} with $V_x \cap Cl(R_{Vx}) = \emptyset$ and a λ-open set U_x containing x such that $f(U_x) \subset V_x \cup R_{Vx}$. Then there exists a finite subfamily $\{U_x\}_{x \in \Delta}$ which covers A, where Δ is a finite subset of A. Moreover, $\{V_x \cup R_{Vx}\}_{x \in \Delta}$ covers f(A). Thus, f(A) is rarely almost compact subset relative to Y.

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دراسة نوع جديد من أشكال ضعف بوبا النادرة الاستمرارية خلال المجموعات المفتوحة – λ

إردال إكيسي و **سعيد جعفري

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خلاصة

قدم بوبا في عام 1979م، مفهوم الاستمرارية النادرة باعتبارها تعميم ضعيف الاستمرارية. إن الهدف من هذه الورقة هو تقديم تعميم جديد للاستمرارية النادرة الذي يدعى استمرارية – λ بين الاستمرارية والتحقيق في بعض الخصائص الأساسية.