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ON UNIVERSALLY MAXIMAL IDEALS

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Abstract. The purpose of this paper is to find necessary and sufficient conditions for a semi-group S to possess a universally maximal two-sided (left, right) ideal. The notion of a "basic element" is introduced in this paper to use it in obtaining these necessary and sufficient conditions.

DEFINITIONS

We begin by introducing some definitions which may be found in (Clifford and Preston 1961) or in the other two references at the end of the paper.

A two-sided (left, right) ideal A of a semi-group S is called "MAXIMAL" if : $A \neq S$ and is not contained properly in any proper two-sided (left, right) ideal of S.

A left (right, two-sided) ideal $L^*(R^*, T^*)$ of a semigroup S is called "UNIVERSALLY MAXIMAL" if: $L^* \neq S$ $(R^* \neq S$, $T^* \neq S)$ and contains every proper left (right, two-sided) ideal of S.

An element a of a semigroup S is called a "UNIVERSAL" left (right, interior) divisor of S if: $aS = S$ $(Sa = S$, $SaS = S)$.

An element a of a semigroup S is called a right (left, two-sided) "BASIC" element of S if: $a \cup Sa = S$ $\quad (a \cup aS = S$, $a \cup Sa \cup aS \cup SaS = S)$.

NOTATIONS

In a semigroup S:

$\mathscr{A} = \{a \in S : aS \cup a = S \text{ and } aS \neq S\}$

$\mathscr{B} = \{a \in S : a \cup Sa = S \text{ and } Sa \neq S\}$

$\mathscr{D} = \{a \in S : a \cup Sa \cup aS \cup SaS = S \text{ and } Sa \cup aS \cup SaS \neq S\}$

$\mathscr{E} = \{a \in S : aS = S \text{ and } Sa \neq S\}$

$\mathscr{F} = \{a \in S : Sa = S \text{ and } aS \neq S\}$

$\mathscr{G} = \{a \in S : Sa = aS = S\}$

$A = \{a \in S : a \cup aS = S\}$

$B = \{a \in S : a \cup Sa = S\}$

$D = \{a \in S : a \cup Sa \cup aS \cup SaS = S\}$

REMARK

Obviously, $A = \mathscr{A} \cup \mathscr{G} \cup \mathscr{E}$

$B = \mathscr{B} \cup \mathscr{F} \cup \mathscr{G}$

$D \supseteq \mathscr{D} \cup \mathscr{E} \cup \quad \cup \mathscr{G}$

Theorem 1

Each of $\mathscr{A}, \mathscr{B}, \mathscr{D}$ is either empty or has only one element.

Proof :

Let $\mathscr{A} \neq \phi$ then there exists $a \in S$ such that $a \cup aS = S$ and $aS \neq S$.

Since aS is a right ideal of S, then for every $x \in aS$ (that is , for every $x \in S$ such that $x \neq a$) we have : $x \cup xS \subseteq aS \subset S$.

Hence $\mathscr{A} = \{a\}$. The proofs for $\mathscr{B}$ and $\mathscr{D}$ are analogous.

Theorem 2

(1) If $\mathscr{A} \ddagger \phi$ then $\mathscr{E} = \mathscr{G} = \phi$

(2) If $\mathscr{B} \ddagger \phi$ then $\mathscr{F} = \mathscr{G} = \phi$

(3) If $\mathscr{D} \ddagger \phi$ then $\mathscr{E} = \mathscr{F} = \mathscr{G} = \phi$

Proof :

(1) If $\mathscr{A} \ddagger \phi$ then there exists only one element $a \in S$ such that $a \cup aS = S$ and $aS \neq S$.
Hence for every $x \in S$, $xS \neq S$.
Therefore $\mathscr{E} = \mathscr{G} = \phi$
Evidently if $\mathscr{E} \ddagger \phi$ or $\mathscr{G} \ddagger \phi$ then $\mathscr{A} = \phi$

(2) If $\mathscr{D} \ddagger \phi$ then S has only one element a such that $a \cup Sa \cup aS \cup SaS = S$ and $aS \cup Sa \cup SaS \neq S$.

Hence for every $x \in S$ such that $x \neq a$, $x \cup Sx \cup xS \cup SxS \subseteq Sa \cup aS \cup SaS \subset S$
Therefore $xS \neq S$ and $Sx \neq S$
that is, $\mathscr{E} = \mathscr{F} = \mathscr{G} = \phi$

REMARK

In a semigroup S, it can be $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{D}$ are not empty ; the following example illustrates that, since $\mathscr{A} = \mathscr{B} = \mathscr{D} = \{a\}$

	a	b	c	d
a	a	b	c	b
b	b	c	a	c
c	c	a	b	a
b	b	c	a	c

COROLLARY

We can see easily that exactly one of the following statements must hold for a semigroup S with regard to left (right) basic elements:

(1) S has no left (right) basic elements.

(2) S has only one left (right) basic element but no universal left (right) divisor of S.

(3) S has one or more left (right) basic elements which are all universal left (right) divisors of S. Similarly, we may state that the following, for two-sided basic elements :

(1) S has no two-sided basic elements.

(2) S has only one two-sided basic element, but no universal left, no right, and no interior divisor of S.

(3) S has one or more two-sided basic elements, but for each x of these basic elements $Sx \cup xS \cup SxS = S$

Theorem 3

If a semigroup S has exactly one universal left (right) divisor a, then a is a left (right) identity of S.

Proof :

a is a universal left divisor of S, then $aS = S$, hence $a^2S = aS = S$, therefore $a^2 = a$.

For every $y \in S$ there exists $x \in S$ such that $y = ax$ then $ay = a^2x = ax = y$

COROLLARY

If a semigroup S has exactly one universal left divisor a, and exactly one universal right divisor b, then S has an identity element

$$e = a = b.$$

REMARK

The converse of Theorem 3 is not true, also for the above corollary.

The following example illustrates that:

	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	c	d	e	f
c	a	c	e	a	c	e
d	a	d	a	d	a	d
e	a	e	c	a	e	c
f	a	f	e	d	c	b

Theorem 4

A semigroup S has the universally maximal two-sided (left, right) ideal $T^*(L^*, R^*)$ if and only if : S has at least one - but not all - two-sided (left, right) basic element. Moreover, T^* (L^*, R^*) is the complement of the set of all two-sided (left, right) basic elements of S.

Proof:

(1) S has a two-sided basic element a, but not all.
Let $D = \{x \in S: x \cup Sx \cup xS \cup SxS = S\}$
Since $D \neq \phi$ and $D \neq S$ then the complement $\bar{D} \neq S$ and $\bar{D} \neq \phi$.
$\bar{D}$ is a two-sided ideal of S, since for every $x \in S$, and for every $y \in \bar{D}$:

(i) $xy \cup Sxy \cup SxyS \subseteq Sy \cup SyS \subseteq y \cup Sy \cup yS \cup SyS \subset S$

(ii) $yx \cup Syx \cup yxS \cup SyxS \subseteq yS \cup SyS \subseteq y \cup Sy \cup yS \cup SyS \subset S$

Hence xy and yx in $\bar{D}$.
That is, $S\bar{D} \subseteq \bar{D}$ and $\bar{D}S \subseteq \bar{D}$.
Now, let T be any two-sided ideal of S, such that $T \nsubseteq \bar{D}$ then $T \cap D \neq \phi$. Hence, there exists an element $b \in T$ such that $b \cup Sb \cup bS \cup SbS = S$. But $b \in T$ implies that $b \cup Sb \cup bS \cup SbS \subseteq T$, therefore $S \subseteq T$, but $T \subseteq S$, then $T = S$.

(2) S has T^*.

Hence there exists an element $h \in S$ such that $h \notin T^*$.

Since $h \cup Sh \cup hS \cup ShS = T$ is a two-sided ideal of S, then $T \subseteq T^*$ or $T = S$

But $h \notin T^*$, therefore $T = S$

Hence S has a two-sided basic element h.

References

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حول المثل العظمى عموميا

أنور توفيق اللحام

قسم الرياضيات بجامعة الكويت

خلاصـــة

لقد تم ايجاد الشروط الضرورية والكافية لكي تمتلك نصف الزمرة مثلا أعظـم عموما وذا جانبين (أو أيسر أو أيمن) . وأدخلت مفهومة العنصر الاساسي من أجل الحصول على هذه الشروط .