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The classical torsion problem for certain curvilinear and sectorial cross sections

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ABSTRACT

This is a continuation of our previous paper which dealt with solutions of Saint-Venant's torsion problem for certain simply connected cross sections. The symmetric cross sections considered here are either curvilinear sections bounded by hyperbolic arcs of certain eccentricities or sectorial sections bounded by two straight lines and special curved bases. Exact closed expressions are established for the stress functions, torsional rigidities and shearing stresses. Numerical results are presented in the form of tables and graphs.

1. INTRODUCTION

The literature of the classical Saint-Venant's torsion problem is very extensive. Some references dealing with the subject are to be found in our previous paper (Bassali & Obaid 1981). A comprehensive bibliography dealing with the theory of elastic cylinders in torsion and including conventional as well as complex variable methods is given by Chien *et al.* (1956), which was reviewed in Appl. Mech. Reviews (1958, p. 110). Another specialised text by Weber & Günther (1958) contains a number of torsion solutions useful in engineering applications. Since the mathematical equations expressing the classical torsion problem are formally the same as those defining boundary value problems in other physical domains, it follows that known torsion solutions are of considerable help in investigating various problems of technical importance (such as certain structural problems in the design of modern high-speed aircraft and many hydrodynamical analogies). In addition to the various cross sections studied in our previous paper (Bassali & Obaid 1981), two more families of sections are considered in this paper, each family containing a certain parameter. Members of the first family are bounded by symmetric curvilinear hyperbolic arcs of certain eccentricities while the second family consists of uniaxial symmetric cross sections bounded by two linear sides and certain curved bases. The shapes of some sections corresponding to particular values of the parameters are sketched. Closed and exact expressions are established for the torsional rigidities and shearing stresses and their variation with the parameters involved is exhibited both numerically and graphically.

2. FUNDAMENTAL EQUATIONS

We consider an elastic isotropic homogeneous cylinder of uniform simply connected cross section S bounded by a simple closed curve Γ in the z-plane $(z=x+iy=re^{i\theta})$ which is chosen to coincide with the plane $Z=0$ perpendicular to the generators of the cylinder. We assume that the Cartesian equation of the closed boundary Γ has the general form

$$Re\,F(z)\;-\;(\alpha x^2+\beta y^2+\gamma)\;=\;0, \tag{1}$$

where $F(z)$ is an arbitrary function of z which is analytic in the domain S enclosed by Γ, and α, β, γ are real constants subject to the condition

$$\alpha+\beta\;\neq\;0. \tag{2}$$

It is well known that the stress function ψ which solves the torsion problem satisfies Poisson's equation

$$\nabla^2\psi\;=\;-2 \tag{3}$$

at any point of S and the boundary condition

$$\psi\;=\;0 \text{ on } \Gamma. \tag{4}$$

It was shown (Bassali & Obaid 1981) that

$$\psi\;=\;\frac{1}{\alpha+\beta}\,[Re\;F(z)-(\alpha x^2+\beta y^2+\gamma)]. \tag{5}$$

This value for ψ is proportional to the expression appearing in equation (1) for the boundary Γ of the cross section. Special cases in which this proportionality holds were considered by Leibenson (1934).

The torsional rigidity D is given by

$$D\;=\;2\mu\iint_S\psi\,r\,dr\,d\theta, \tag{6}$$

and the stress components at any point (r,θ) are determined by

$$\widehat{rZ}\;=\;\frac{\mu T}{r}\frac{\partial\psi}{\partial\theta},\qquad \widehat{\theta Z}\;=\;-\mu T\frac{\partial\psi}{\partial r}, \tag{7}$$

where μ is the modulus of rigidity of the material of the cylinder and T is the constant twist per unit length.

3. CROSS SECTIONS BOUNDED BY HYPERBOLIC ARCS

Taking $F(z)=z^4$ in (1) we find that the Cartesian equation of Γ is

$$x^4-6x^2y^2+y^4-\alpha x^2-\beta y^2-\gamma\;=\;0. \tag{8}$$

Setting $\alpha=\beta=\lambda^{-1},\gamma=1-\lambda^{-1}$ where λ is a parameter $(\lambda\neq 0)$, we see that the equation of Γ is

$$\lambda(x^4-6x^2y^2+y^4-1)-x^2-y^2+1\;=\;0, \tag{9}$$

and the corresponding stress function (5) is

$$\psi(x,y)\;=\;\tfrac{1}{2}[\lambda(x^4-6x^2y^2+y^4-1)-x^2-y^2+1]. \tag{10}$$

By varying λ, Saint-Venant obtained the family of cross sections shown in Timoshenko & Goodier (1970, Fig. 156a, p. 301). Any member of this family is symmetric with respect to both axes and passes through the four points $(\pm 1, 0)$ and $(0, \pm 1)$.

In this section we choose values for α, β and γ such that the fourth degree polynomial in (8) can be factorised to two quadratic factors. Taking

$$\alpha = -4c^2, \qquad \beta = \lambda\alpha, \qquad \gamma = \tfrac{1}{2}(1+6\lambda+\lambda^2)c^2, \tag{11}$$

where c is a linear constant and λ is a parameter, we easily verify that (8) may be written as

$$\left\{x^2-(3+2\sqrt{2})y^2+\left(\frac{3+\lambda}{\sqrt{2}}+2\right)c^2\right\}\left\{x^2-(3-2\sqrt{2})y^2-\left(\frac{3+\lambda}{\sqrt{2}}-2\right)c^2\right\} = 0 \tag{12}$$

and the cross section S is thus bounded by the two hyperbolas

$$(3+2\sqrt{2})y^2-x^2 = \left(\frac{3+\lambda}{\sqrt{2}}+2\right)c^2, \tag{13a}$$

$$x^2-(3-2\sqrt{2})y^2 = \left(\frac{3+\lambda}{\sqrt{2}}-2\right)c^2, \tag{13b}$$

which intersect in the four points $(\pm a, \pm b)$ where

$$a = \tfrac{1}{2}c\sqrt{(1+3\lambda)}, \qquad b = \tfrac{1}{2}c\sqrt{(3+\lambda)}. \tag{14}$$

The corresponding stress function (5) is

$$\psi(r,\theta) = \frac{1}{8(1+\lambda)c^2}\left[(1+6\lambda+\lambda^2)c^2-4\{1+\lambda+(1-\lambda)\cos 2\theta\}c^2r^2-2r^4\cos 4\theta\right]. \tag{15}$$

The asymptotes of (13a) are

$$y = \pm(\sqrt{2}-1)x = \pm x\tan \pi/8$$

and those of (13b) are

$$y = \pm(\sqrt{2}+1)x = \pm x\tan 3\pi/8.$$

For convenience we introduce the two parameters k and ϕ defined by

$$k = \frac{a}{b} = \cot\phi = \sqrt{\frac{1+3\lambda}{3+\lambda}}. \tag{16}$$

For the boundary Γ of the cross section to be a curvilinear rectangle, the right side of (13b) must be positive and we have the restriction

$$2\sqrt{2}-3 < \lambda < \infty \tag{17}$$

but this range will now be reduced without loss of generality. If this condition is satisfied then the eccentricity e_1 of either hyperbola equals $\sec 3\pi/8 = \sqrt{(4+2\sqrt{2})} = 2{\cdot}6131$. The cross section corresponding to $\lambda = 0$ is shown in Fig. 1 and in this case Equation (16) gives

$$k = 1/\sqrt{3}, \qquad \phi = \pi/3. \tag{18}$$

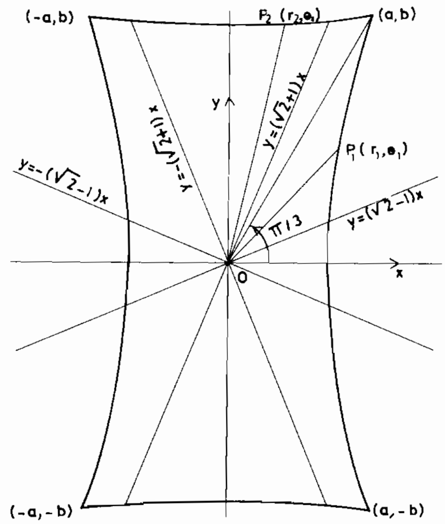

Fig. 1. A curvilinear rectangle ($\lambda=0$, $k=1/\sqrt{3}=0{\cdot}577$).

If $(\pm a_1, \pm b_1)$ are the four vertices of the curvilinear rectangle corresponding to the values (λ_1, c_1) and $(\pm a_2, \pm b_2)$ are the four vertices corresponding to (λ_2, c_2), then (14) gives

$$a_v = \tfrac{1}{2}c_v\sqrt{(1+3\lambda_v)}, \qquad b_v = \tfrac{1}{2}c_v\sqrt{(3+\lambda_v)} \quad (v = 1,2).$$

Setting $a_1=b_2$ and $a_2=b_1$ easily leads to

$$\lambda_1\lambda_2 = 1, \qquad c_1/c_2 = \sqrt{\lambda_2} = 1/\sqrt{\lambda_1}. \tag{19}$$

This shows that the range (17) can be reduced to

$$2\sqrt{2}-3 < \lambda \le 1, \tag{20}$$

and applying (16) we deduce that

$$\sqrt{2}-1 < k \le 1, \qquad 3\pi/8 > \phi \ge \pi/4. \tag{21}$$

The polar equations of the hyperbolas (13a,b) are

$$r^2 = c^2A/(\sqrt{2}+2\cos 2\theta), \tag{22a}$$

$$r^2 = c^2B/(\sqrt{2}-2\cos 2\theta), \tag{22b}$$

where

$$A = \tan(\pi/8)+\lambda\tan(3\pi/8), \qquad B = \tan(3\pi/8)+\lambda\tan(\pi/8). \tag{23}$$

For $\lambda=1$, $a=b=c$, $k=1$, $\phi=\pi/4$ and we have the case of the curvilinear square bounded by the two hyperbolas

$$r^2 = 2c^2/(1\pm\sqrt{2}\cos 2\theta), \tag{24a}$$

which intersect in the four points $(\pm c, \pm c)$.

In the special case $\lambda=2\sqrt{2}-3$, $k=\sqrt{2}-1$, $\phi=3\pi/8$ the hyperbola (13a) degenerates to the two straight lines $y=\pm x\tan(3\pi/8)$ and we obtain the sectorial section drawn in Fig. 2 and bounded by these lines and the hyperbola

$$r^2 = 4(\sqrt{2}-1)c^2/(1-\sqrt{2}\cos 2\theta). \tag{24b}$$

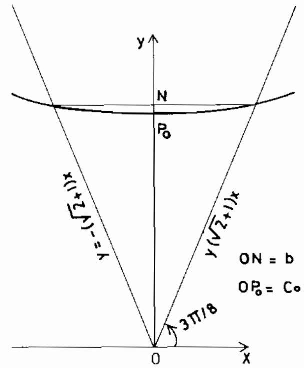

Fig. 2. Sectorial cross section with an included angle of $\pi/4$ and base a certain hyperbola.

In Fig. 2 it is easily seen that

$$ON = b = c/2^{1/4}, \qquad OP_0 = c_0 = 2(\sqrt{2}-1)c, \qquad b/c_0 = 1{\cdot}0151. \quad (25)$$

For $-1/3<\lambda<2\sqrt{2}-3$, $0<k<\sqrt{2}-1$, $3\pi/8<\phi<\pi/2$ we have the case of a crescent-like cross section bounded by the two hyperbolas (13a,b) with eccentricities e_1 and $e_2=\sec(\pi/8)=\sqrt{(4-2\sqrt{2})}=1{\cdot}0824$, respectively. The cross section corresponding to $\lambda=-\frac{1}{4}$ is shown in Fig. 3.

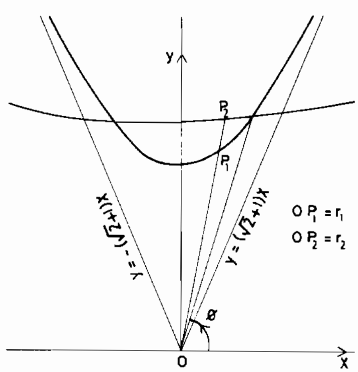

Fig. 3. A crescent-like cross section ($\lambda=-1/4$, $k=1/\sqrt{11}=0{\cdot}302$).

We now turn our attention to the determination of the torsional rigidities of the cross sections. For the curvilinear rectangle (Fig. 1) substitution from (15) in (6) yields

$$D = \frac{\mu}{(1+\lambda)c^2}\left[\int_0^{\phi}\left\{\tfrac{1}{2}(1+6\lambda+\lambda^2)c^4r_1^2-(1+\lambda+\overline{1-\lambda}\cos 2\theta)c^2r_1^4-\frac{1}{3}r_1^6\cos 4\theta\right\}d\theta+\right.$$
$$\left.+\int_{\phi}^{\pi/2}\left\{\tfrac{1}{2}(1+6\lambda+\lambda^2)c^4r_2^2-(1+\lambda+\overline{1-\lambda}\cos 2\theta)c^4r_2^4-\frac{1}{3}r_2^b\cos 4\theta\right\}d\theta\right],$$

where r_1, r_2 are given by (22a,b), respectively. Thus we get

$$D = \mu c^4(A_1I_1+A_2I_2+B_1J_1+B_2J_2)/(1+\lambda),$$

where

$$A_1 = \frac{A\sqrt{2}}{4}\left\{1+6\lambda+\lambda^2-(1-\lambda)A-\frac{1}{3}A^2\right\}, \qquad A_2 = A^2\left\{\frac{\sqrt{2}}{4}(1-\lambda)-\tfrac{1}{2}(1+\lambda)+\frac{\sqrt{2}A}{6}\right\},$$

$$B_1 = \frac{B\sqrt{2}}{4}\left\{1+6\lambda+\lambda^2+(1-\lambda)B-\frac{1}{3}B^2\right\}, \qquad B_2 = B^2\left\{-\frac{\sqrt{2}}{4}(1-\lambda)-\tfrac{1}{2}(1+\lambda)+\frac{\sqrt{2}B}{6}\right\},$$

$$I_n = \int_0^{\phi}(1+\sqrt{2}\cos 2\theta)^{-n}d\theta, \qquad J_n = \int_{\phi}^{\pi/2}(1-\sqrt{2}\cos 2\theta)^{-n}d\theta\,(n = 1,2).$$

After evaluating these integrals and performing some algebraic manipulation we arrive at the formula

$$D_{\mathrm{curv}}(c,\lambda) = \tfrac{1}{4}\mu c^4\left[\{\sqrt{2}-1+(\sqrt{2}+1)\lambda\}^2 \ln\left|\frac{(\sqrt{2}+1)\sqrt{1+3\lambda}+\sqrt{3+\lambda}}{(\sqrt{2}+1)\sqrt{1+3\lambda}-\sqrt{3+\lambda}}\right| + \right.$$
$$+\{\sqrt{2}+1+(\sqrt{2}-1)\lambda\}^2 \ln\left|\frac{(\sqrt{2}+1)\sqrt{3+\lambda}+\sqrt{1+3\lambda}}{(\sqrt{2}+1)\sqrt{3+\lambda}-\sqrt{1+3\lambda}}\right| -$$
$$\left. -\frac{1+2\lambda/3+\lambda^2}{1+\lambda}\sqrt{2(1+3\lambda)(3+\lambda)}\right]. \tag{26}$$

Using (19) it is easily verified that $D_{\mathrm{curv}}(c_1,\lambda_1)=D_{\mathrm{curv}}(c_2,\lambda_2)$ as it should. The torsional rigidity of the curvilinear square bounded by the two hyperbolas (24a) is

$$D_{\mathrm{curv}}(c,1) = 4\mu c^4\left[\ln(\sqrt{2}+1)-\frac{\sqrt{2}}{3}\right] = 1{\cdot}6399\mu c^4(a = b = c), \tag{27a}$$

and the torsional rigidity of the curvilinear rectangle in Fig. 1 which corresponds to $\lambda=0$ is

$$D_{\mathrm{curv}}(c,0) = \tfrac{1}{4}\mu c^4[6\ln(\sqrt{2}+\sqrt{3})-2\sqrt{2}\ln(2+\sqrt{3})-\sqrt{6}]$$
$$= 0{\cdot}1757\mu c^4\left(a = \tfrac{1}{2}c, b = \frac{\sqrt{3}}{2}c\right). \tag{27b}$$

Applying (14) and (16) we find that the formula (26) can be written in the equivalent forms

$$D_{\mathrm{curv}}(c,\phi) = \tfrac{1}{4}\mu c^4\left[\left(\tan\frac{\pi}{8}+\lambda\tan\frac{3\pi}{8}\right)^2 \ln\left|\frac{\sin(3\pi/8+\phi)}{\sin(3\pi/8-\phi)}\right| + \right.$$
$$+\left(\tan\frac{3\pi}{8}+\lambda\tan\frac{\pi}{8}\right)^2 \ln\left|\frac{\sin(\phi+\pi/8)}{\sin(\phi-\pi/8)}\right| -$$
$$\left. -2\sqrt{2}\left(1+\frac{2}{3}\lambda+\lambda^2\right)\sin 2\phi\right]\left(\lambda = -\frac{\tan 3\phi}{\tan\phi}\right), \tag{28}$$

$$D_{\mathrm{curv}}(b,k) = \tfrac{1}{2}\mu b^4\left[\{(\sqrt{2}+1)k^2-\sqrt{2}+1\}^2 \ln\left|\frac{(\sqrt{2}+1)k+1}{(\sqrt{2}+1)k-1}\right| + \right.$$
$$+\{\sqrt{2}+1-(\sqrt{2}-1)k^2\}^2 \ln\left|\frac{\sqrt{2}+1+\mathrm{k}}{\sqrt{2}+1-\mathrm{k}}\right| -$$
$$\left. -\frac{4\sqrt{2}k(1-2k^2/3+k^4)}{1+k^2}\right](\sqrt{2}-1 < k \le 1). \tag{29a}$$

Table 1

k	$D_{\text{curv}}/\mu b^4$	$D_{\text{rect}}/\mu b^4$	$D_{\text{curv}}/D_{\text{rect}}$
0·42	0·0412	0·2992	0·1376
0·44	0·0619	0·3387	0·1826
0·46	0·0875	0·3810	0·2295
0·48	0·1173	0·4261	0·2752
0·50	0·1509	0·4739	0·3184
0·52	0·1881	0·5244	0·3587
0·54	0·2286	0·5776	0·3958
0·56	0·2722	0·6333	0·4298
0·58	0·3187	0·6916	0·4608
0·60	0·3679	0·7523	0·4891
0·62	0·4196	0·8154	0·5147
0·64	0·4737	0·8807	0·5379
0·66	0·5299	0·9481	0·5589
0·68	0·5880	1·0175	0·5779
0·70	0·6479	1·0888	0·5950
0·72	0·7094	1·1618	0·6106
0·74	0·7723	1·2364	0·6246
0·76	0·8364	1·3123	0·6373
0·78	0·9015	1·3893	0·6489
0·80	0·9675	1·4674	0·6593
0·82	1·0342	1·5462	0·6689
0·84	1·1015	1·6256	0·6776
0·86	1·1691	1·7053	0·6856
0·88	1·2369	1·7850	0·6930
0·90	1·3048	1·8645	0·6998
0·92	1·3726	1·9436	0·7062
0·94	1·4401	2·0219	0·7123
0·96	1·5073	2·0991	0·7181
0·98	1·5739	2·1750	0·7236
1	1·6399	2·2492	0·7291

The exact and closed expression (29a) for the torsional rigidity of the curvilinear rectangle may be compared with the following value (Sokolnikoff 1956, p. 132) for the rectangular cross section with sides $2a$, $2b$:

$$D_{\text{rect}}(b,k) = 16\mu b^4\left[\frac{1}{3}k^3 - \frac{64k^4}{\pi^5}\sum_{n=0}^{\infty}\frac{\tanh\,[(2n+1)\pi/2k]}{(2n+1)^5}\right](k = a/b). \quad (29b)$$

Numerical values calculated from (29a,b) are listed in Table 1 and plotted in Fig. 4.

The torsional rigidity of the crescent-like cross section (Fig. 3) is furnished by

$$D_{\text{cres}}(c,\lambda) = \frac{\mu}{(1+\lambda)c^2}\int_{\phi}^{\pi/2}\int_{r_1}^{r_2}[\tfrac{1}{2}(1+6\lambda+\lambda^2)c^4 - 2\{1+\lambda+(1-\lambda)\cos 2\theta\}c^2r^2 - r^4\cos 4\theta]\,r\,dr\,d\theta,$$

where ϕ is defined by (16), r_1, r_2 are given by (22a,b) and $-1/3 < \lambda < 2\sqrt{2}-3$. It is found that the resulting value for $D_{\text{cres}}(c,\lambda)$ equals half the value given by (26) or (28).

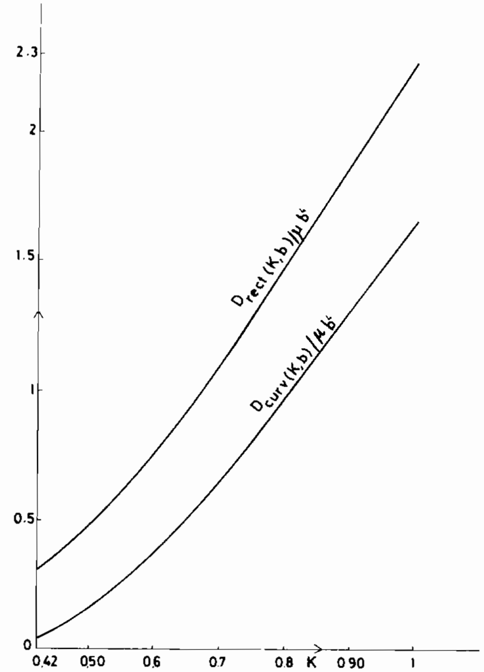

Fig. 4. Variation of the torsional rigidities of rectangular and curvilinear cross sections with k $(0{\cdot}42 \leq k \leq 1)$.

Table 2

k	$10^5 D_{\text{cres}}(k,b)/\mu b^4$	k	$10^5 D_{\text{cres}}(k,b)/\mu b^4$
0·20	9	0·32	255
0·22	18	0·34	397
0·24	33	0·36	606
0·26	58	0·38	912
0·28	98	0·40	1361
0·30	160	$\sqrt{2}-1=0{\cdot}414$	1817

Similarly $D_{\text{cres}}(b,k)$ equals half the value given by (29a), where $0<k<\sqrt{2}-1$. Numerical values are given in Table 2 and graphically plotted in Fig. 5.

The torsional rigidity of the sectorial cross section in Fig. 2 is found by taking half the value obtained by putting $k=\sqrt{2}-1$ in (29a). Thus we get

$$D = (4/3)(3-2\sqrt{2})(3 \ln 2-2)\mu b^4 = 0{\cdot}01817\ \mu b^4. \tag{30}$$

We now study the distribution of shearing stress for the curvilinear rectangular cross section (Fig. 1). Substituting from (15) in (7) gives

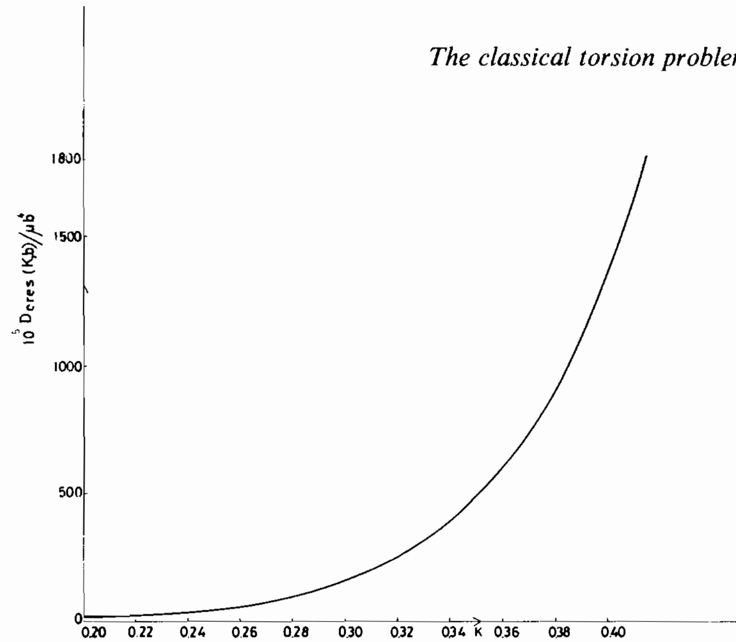

Fig. 5. Variation of the torsional rigidity of crescent-like cross sections with k $(0{\cdot}2 \leq k \leq \sqrt{2}-1)$.

$$\widehat{rZ} = \frac{\mu Tr}{1+\lambda}\left[(1-\lambda)\sin 2\theta + \frac{r^2}{c^2}\sin 4\theta\right], \tag{31a}$$

$$\widehat{\theta Z} = \frac{\mu Tr}{1+\lambda}\left[1+\lambda+(1-\lambda)\cos 2\theta + \frac{r^2}{c^2}\cos 4\theta\right], \tag{31b}$$

$$\sigma(r,\theta) = \sqrt{[(\widehat{rZ})^2+(\widehat{\theta Z})^2]} =$$

$$\frac{2\mu Tr}{1+\lambda}\left[\cos^2\theta+\lambda^2\sin^2\theta+(\cos\theta\cos 3\theta-\lambda\sin\theta\sin 3\theta)\frac{r^2}{c^2}+\frac{r^4}{4c^4}\right]^{1/2}. \tag{31c}$$

At points on the axes of symmetry we have

$$\sigma(r,0) = \frac{\mu Tr}{1+\lambda}\left(2+\frac{r^2}{c^2}\right), \qquad \sigma\left(r,\frac{\pi}{2}\right) = \frac{\mu Tr}{1+\lambda}\left(2\lambda+\frac{r^2}{c^2}\right), \tag{32}$$

while at points of Γ we have (Fig. 1)

$$\sigma(r_1,\theta_1) = \frac{\mu Tr_1}{1+\lambda}\left|1-\lambda+2(1+\lambda)\cos 2\theta_1\right|\frac{\sqrt{(3+2\sqrt{2}\cos 2\theta_1)}}{\sqrt{2}+2\cos 2\theta_1}(0 \leq \theta_1 \leq \phi), \tag{33a}$$

$$\sigma(r_2,\theta_2) = \frac{\mu Tr_2}{1+\lambda}\left|1-\lambda+2(1+\lambda)\cos 2\theta_2\right|\frac{\sqrt{(3-2\sqrt{2}\cos 2\theta_2)}}{\sqrt{2}-2\cos 2\theta_2}\left(\phi \leq \theta_2 \leq \frac{\pi}{2}\right). \tag{33b}$$

The distribution of the peripheral shearing stress along the edges of the curvilinear rectangle corresponding to $\lambda = 0$ (Fig. 1) and the curvilinear square ($\lambda = 1$) is exhibited in Tables 3 and 4 and Figs 6 and 7.

Table 3

θ_1	$\sigma(r_1,\theta_1)/\mu Ta$	θ_2	$\sigma(r_2,\theta_2)/\mu TA$
0°	$1{\cdot}4778 = 3(\sqrt{2}-1)2^{1/4}$	60°	0
3°	1·4775	61·5°	0·1492
6°	1·4768	63°	0·2827
9°	1·4755	64·5°	0·4021
12°	1·4736	66°	0·5092
15°	1·4710	67·5°	0·6051
18°	1·4677	69°	0·6910
21°	1·4634	70·5°	0·7678
24°	1·4579	72°	0·8364
27°	1·4509	73·5°	0·8975
30°	1·4419	75°	0·9516
33°	1·4303	76·5°	0·9992
36°	1·4150	78°	1·0408
39°	1·3944	79·5°	1·0767
42°	1·3661	81°	1·1073
43·5°	1·3477	82·5°	1·1327
45°	1·3257	84°	1·1533
46·5°	1·2987	85·5°	1·1691
48°	1·2655	87°	1·1803
49·5°	1·2238	88·5°	1·1870
51°	1·1706	90°	$1{\cdot}1892 = 2^{1/4}$
52·5°	1·1014		
54°	1·0091		
55·5°	0·8822		
57°	0·7013		
58·5°	0·4308		
60°	0		

Table 4

θ_1	$\sigma(r_1,\theta_1)/\mu Ta$	θ_1	$\sigma(r_1,\theta_1)/\mu Ta$
0°	$1{\cdot}2872 = 2\sqrt{(\sqrt{2}-1)}$	24·75°	1·0751
2·25°	1·2857	27°	1·0243
4·5°	1·2812	29·25°	0·9643
6·75°	1·2740	31·5°	0·8932
9°	1·2635	33·75°	0·8082
11·25°	1·2497	36°	0·7062
13·5°	1·2323	38·25°	0·5823
15·75°	1·2109	40·5°	0·4301
18°	1·1851	42·75°	0·2405
20·25°	1·1544	45°	0
22·5°	1·1180		

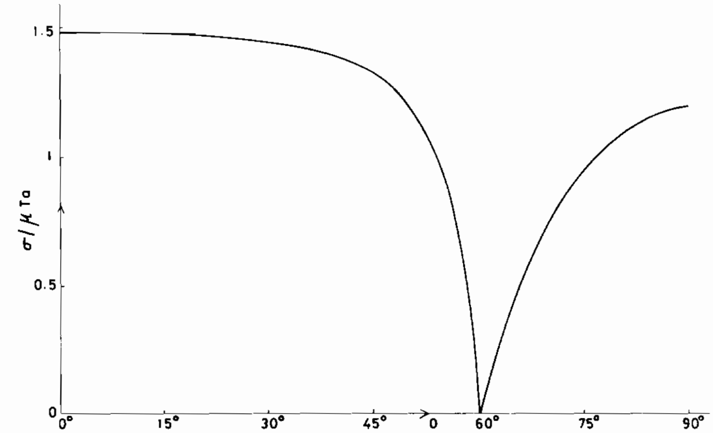

Fig. 6. Variation of peripheral shearing stress along the boundary of the curvilinear rectangle ($\lambda=0$).

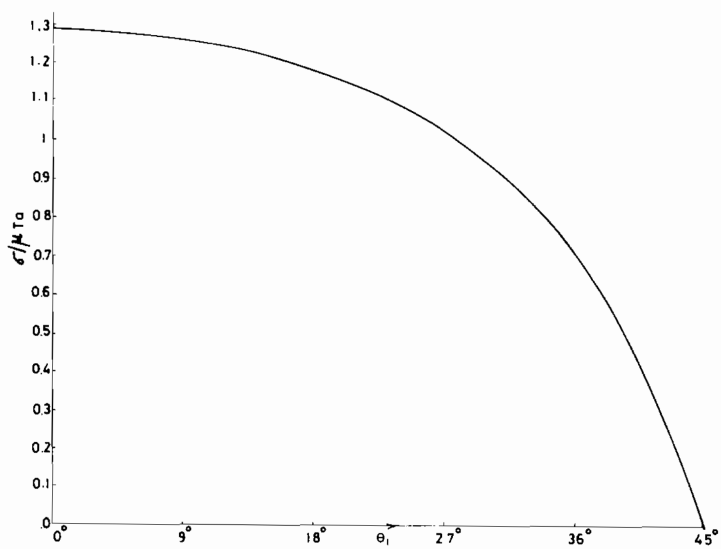

Fig. 7. Variation of peripheral shearing stress on the boundary of the curvilinear square ($\lambda=1$).

4. SECTORIAL CROSS SECTIONS WITH TWO LINEAR SIDES AND CERTAIN CURVED BASES

In this section we solve the torsion problem for the family of cross sections bounded by Γ_n which consists of the two straight lines $\theta=\pm\pi/2n$ and the curved base

$$(r/c_0)^{n-2}=\frac{\cos 2\theta-\cos\pi/n}{\cos n\theta(1-\cos\pi/n)}(1<n<\infty,-\pi/2n\leq\theta\leq\pi/2n,\ n\neq 2). \quad (34)$$

This curve is symmetric with respect to the initial line $\theta=0$ and cuts it orthogonally in the point $P_0(c_0,0)$. The three members of the family (34) corresponding to $n=3/2$, 3 and

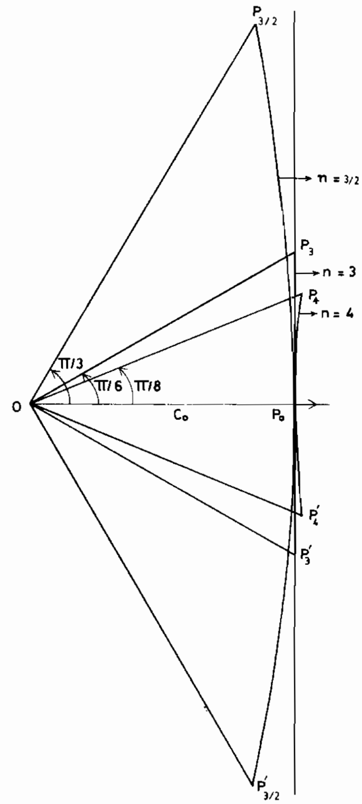

Fig. 8. Sectorial cross sections corresponding to $n=3/2$, 3 and 4.

4 are shown in Fig. 8. The case $n=2$ requires separate consideration. The boundary Γ_2 consists of the two straight lines $\theta=\pm\pi/4$ ($y=\pm x$) and the curve

$$r = c_0 \lim_{n\to 2}\left[\frac{\cos 2\theta-\cos \pi/n}{\cos n\theta(1-\cos \pi/n)}\right]^{1/(n-2)} = c_0 \exp\{\theta \tan 2\theta+(\pi/4)(1-\sec 2\theta)\}. \quad (35)$$

The linear sides intersect the curve base in the two points $P_n(r_n, \pi/2n)$ and $P'_n[r_n, -(\pi/2n)]$ where

$$r_n = c_0 \lim_{\theta\to\pi/2n}\left[\frac{\cos 2\theta-\cos \pi/n}{\cos n\theta(1-\cos \pi/n)}\right]^{1/(n-2)} = c_0\left(\frac{n}{2}\tan\frac{\pi}{2n}\right)^{1/(2-n)} \quad (n \neq 2), \qquad (36a)$$

$$r_2 = \lim_{n\to 2} r_n = c_0 \exp\{(\pi-2)/4\} = 1{\cdot}3303\, c_0. \qquad (36b)$$

The same value for r_2 is obtained by letting θ tend to $\pi/4$ in (35). For $n=3$ equation (34) simplifies to the straight line $x=c_0$ and Γ_3 is the equilateral triangle $OP_3P'_3$ in Fig. 8. For $n=4$ the curved base (34) is the hyperbola

$$x^2-y^2 \tan^2(\pi/8) = c_0^2, \qquad (37)$$

and it can be shown that the cross section bounded by Γ_4 is identical with that of Fig. 2. For $n>3$ it can be proved that the curved base in Fig. 8 lies on the right side of $x=c_0$ while for $n<3$ it lies on the left side of the same line.

If n is an even integer $2m$ then (Jolley 1961)

$$\cos 2m\theta = 2^{m-1} \prod_{\nu=1,3,\ldots}^{2m-1} (\cos 2\theta - \cos \nu\pi/2\mathrm{m})\ (m = 2,3,\ldots),$$

and using the identity

$$\prod_{\nu=1,3,\ldots}^{2m-1} \sin^2 \nu\pi/4\mathrm{m} = 2^{1-2m},$$

we find that the Cartesian equation of the curved base is

$$\prod_{\nu=3,5,\ldots}^{2m-1} (x^2 - y^2 \cot^2 \nu\pi/4m) = c_0^{2m-2}. \tag{38}$$

For $m=2$ we obtain (37) and for $m=3$ we get the quartic curve

$$(x^2-y^2)(x^2-y^2\tan^2 \pi/12) = c_0^4. \tag{39}$$

If n is an odd integer $2m+1$ $(m=1, 2, \ldots)$ then (Jolley 1961)

$$\cos(2m+1)\theta = 2^m \cos\theta \prod_{\nu=1,3,\ldots}^{2m-1} [\cos 2\theta - \cos \nu\pi/(2m+1)]$$

and applying the identity

$$\prod_{\nu=1,3,\ldots}^{2m-1} \sin^2 \nu\pi/(4m+2) = 2^{-2m},$$

we deduce that the Cartesian equation of the curved base is

$$x \prod_{\nu=3,5,\ldots}^{2m-1} [x^2 - y^2 \cot^2 \nu\pi/(4m+2)] = c_0^{2m-1}. \tag{40}$$

For $m=2$ we have the cubic curve

$$x(x^2-y^2\tan^2 \pi/5) = c_0^3, \tag{41}$$

and for $m=3$ we obtain the quintic curve

$$x(x^2-y^2\tan^2 \pi/7)(x^2-y^2\tan^2 2\pi/7) = c_0^5. \tag{42}$$

We now proceed to determine the stress function for the cross section bounded by Γ_n. Let $t=\tan \pi/2n$. In (1) and (5) we assume tentatively that

$$\alpha = -\frac{t^2}{1-t^2}, \qquad \beta = \frac{1}{1-t^2}, \qquad \gamma = 0, \qquad F(z) = C_n z^n (n \neq 2),$$

where C_n is a real constant to be determined. Inserting these values in (5) and transforming to polar coordinates we obtain

$$\psi_n(r,\theta) = (\cos 2\theta - \cos \pi/n)\left[\frac{C_n r^n \cos n\theta}{\cos 2\theta - \cos \pi/n} + \tfrac{1}{2} r^2 \sec \pi/n\right] (1 < n < \infty, n \neq 2).$$

Equating ψ_n to zero gives $\theta = \pm\pi/2n$ and

$$r^{n-2} = -\frac{1}{2C_n}\frac{\sec \pi/n(\cos 2\theta - \cos \pi/n)}{\cos n\theta}.$$

Comparison with (34) now leads to

$$C_n = \tfrac{1}{2}(1-\sec\pi/n)c_0^{2-n}, \qquad F(z) = \tfrac{1}{2}(1-\sec\pi/n)\frac{z^n}{c_0^{n-2}}(1 < n < \infty, n \neq 2),$$

$$\psi_n(r,\theta) = \tfrac{1}{2}r^2[\sec\pi/n(\cos 2\theta-\cos\pi/n)+(1-\sec\pi/n)(r/c_0)^{n-2}\cos n\theta]. \tag{43}$$

Writing this function in the form

$$\psi_n(r,\theta) = \tfrac{1}{2}r^2\left[(r/c_0)^{n-2}\cos n\theta-1+\frac{\cos 2\theta-(r/c_0)^{n-2}\cos n\theta}{\cos\pi/n}\right]$$

and taking the limit as n tends to 2 we get

$$\psi_2(r,\theta) = \lim_{n\to 2}\psi_n(r,\theta) = \tfrac{1}{2}r^2\{\cos 2\theta-1+(4/\pi)(\theta\sin 2\theta-\cos 2\theta\ln r/c_0)\}. \tag{44}$$

This function can also be directly obtained by setting in (5)

$$\alpha=0,\ \beta=1,\ \gamma=0, \qquad F(z) = -(2z^2/\pi)\ln(z/c_0).$$

Substituting from (43) and (44) in (6) and integrating with respect to r we arrive at the following expressions for the torsional rigidities of the cross sections bounded by Γ_n:

$$D_n = \frac{n-2}{2(n+2)}\frac{(2\sin^2\pi/2n)^{4/(2-n)}}{\cos\pi/n}\mu c_0^4\int_0^{\pi/2n}(\cos 2\theta-\cos\pi/n)^{(n+2)/(n-2)}(\cos n\theta)^{4/(2-n)}d\theta$$

$$(1 < n < \infty, n \neq 2), \tag{45a}$$

$$D_2 = (1/4\pi)\,\mu c_0^4\int_0^{\pi/2}\cos u\exp\{2u\tan u+\pi(1-\sec u)\}du = 0{\cdot}1008\,\mu c_0^4. \tag{45b}$$

The definite integral in (45a) can be evaluated in the special cases $n=3,4,6$ and we have

$$D_3 = (1/10)\mu c_0^4\int_0^{\pi/6}\frac{(2\cos 2\theta-1)^5}{\cos^4 3\theta}d\theta = \frac{\sqrt{3}}{45}\mu c_0^4, \tag{46a}$$

in agreement with the value given for the torsional couple by Sokolnikoff (1956, p. 125),

$$D_4 = \frac{1}{6}(3+2\sqrt{2})\mu c_0^4\int_0^{\pi/8}\frac{(\sqrt{2}\cos 2\theta-1)^3}{\cos^2 4\theta}d\theta = \frac{1}{24}(3+2\sqrt{2})(3\ln 2-2)\mu c_0^4, \tag{46b}$$

which agrees with (3) on using (25),

$$D_6 = \frac{1}{12}(3+2\sqrt{3})\mu c_0^4\int_0^{\pi/12}\frac{(2\cos 2\theta-\sqrt{3})^2}{\cos 6\theta)}d\theta = \frac{1}{48}\mu c_0^4(3+2\sqrt{3})\ln\frac{256}{243}$$

$$= 0{\cdot}0070\,\mu c_0^4. \tag{46c}$$

Numerical values of $D_n/\mu c_0^4$ corresponding to $1 < n \leq 10$ are given in Table 5 and plotted in Figs 9 and 10.

The radial and transverse components of the shearing stress at any point (r,θ) are now obtained by substitution from (43) and (44) in (7). For $1 < n < \infty$, $n \neq 2$ we get

$$\widehat{rZ} = \mu Tr\left[\frac{n}{2}\left(\frac{r}{c_0}\right)^{n-2}\left(\sec\frac{\pi}{n}-1\right)\sin n\theta-\sec\frac{\pi}{n}\sin 2\theta\right], \tag{47a}$$

$$\widehat{\theta Z} = \mu Tr\left[1-\cos 2\theta\sec\frac{\pi}{n}+\tfrac{1}{2}n\left(\frac{r}{c_0}\right)^{n-2}\left(\sec\frac{\pi}{n}-1\right)\cos n\theta\right], \tag{47b}$$

Table 5

n	$D_n/\mu c_0^4$	n	$D_n/\mu c_0^4$
1·1	0·8257	2	0·1008
1·2	0·4707	2·1	0·0891
1·3	0·3348	2·2	0·0800
1·4	0·2596	2·3	0·0718
1·5	0·2107	2·4	0·0649
1·6	0·1760	2·5	0·0590
1·7	0·1500	3	0·0384
1·8	0·1297	3·5	0·0266
1·9	0·1135	4	0·0193
1·91	0·1120	4·5	0·0145
1·92	0·1106	5	0·0111
1·93	0·1092	5·5	0·0088
1·94	0·1078	6	0·0070
1·95	0·1065	6·5	0·0057
1·96	0·1052	7	0·0047
1·97	0·1039	7·5	0·0039
1·98	0·1026	8	0·0033
1·99	0·1014	8·5	0·0028
2	0·1008	9	0·0024
		9·5	0·0021
		10	0·0018

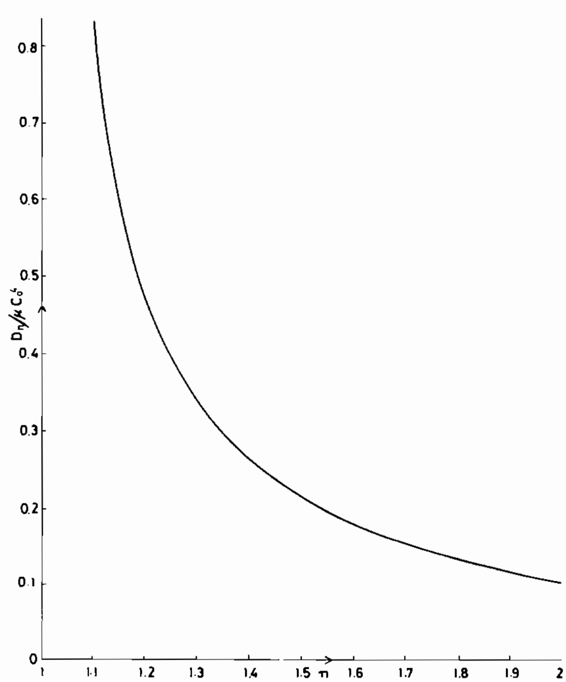

Fig. 9. Variation of torsional rigidities of sectorial sections with n $(1 < n \leq 2)$.

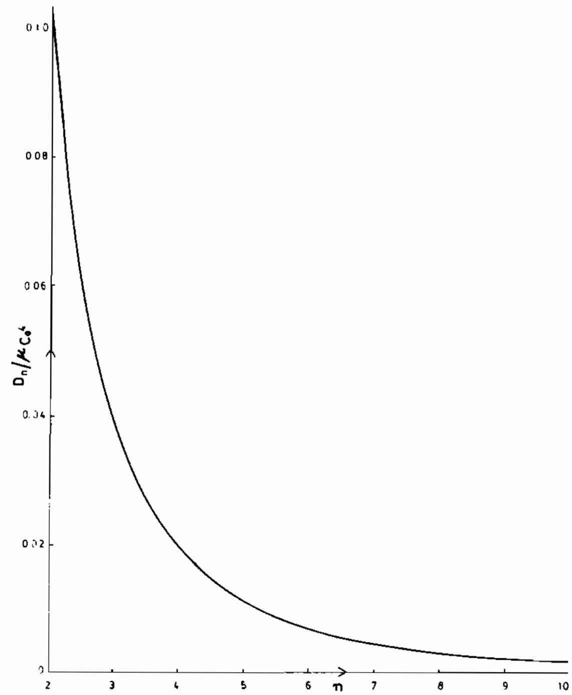

Fig. 10. Variation of torsional rigidities of sectorial sections with n $(2 \leq n \leq 10)$.

$$\sigma_n(r,\theta) = \mu Tr\left[1+\sec^2\frac{\pi}{n}-2\sec\frac{\pi}{n}\cos 2\theta+\frac{1}{4}n^2\left(\sec\frac{\pi}{n}-1\right)^2\left(\frac{r}{c_0}\right)^{2n-4}+\right.$$
$$\left.+n\left(\sec\frac{\pi}{n}-1\right)\left\{\cos n\theta-\sec\frac{\pi}{n}\cos(n-2)\theta\right\}\left(\frac{r}{c_0}\right)^{n-2}\right]^{1/2}. \qquad (47c)$$

On the axis of symmetry OP_0 we have

$$\sigma_n(r,0) = \mu Tr|1-\sec\pi/n| \quad |1-(n/2)(r/c_0)^{n-2}|. \qquad (48)$$

At the point $(r,\pi/2n)$ of OP_n we find

$$\sigma_n(r,\pi/2n) = \mu Tr|\tan(\pi/n)+\tfrac{1}{2}n(1-\sec\pi/n)(r/c_0)^{n-2}|, \qquad (49)$$

and at the point (r,θ) of P_0P_n we obtain

$$\sigma_n(\theta) = \frac{\mu Tr}{2|\cos(\pi/n)|}[(n-2)^2(\cos 2\theta-\cos\pi/n)^2+$$
$$+\{n\tan n\theta(\cos 2\theta-\cos\pi/n)-2\sin 2\theta\}^2]^{1/2}, \qquad (50)$$

where r is given by (34).

The corresponding results for $n=2$ are

$$\widehat{rZ} = \mu Tr[(2/\pi)(\sin 2\theta+2\theta\cos 2\theta+2\sin 2\theta\ln r/c_0)-\sin 2\theta], \qquad (51a)$$
$$\widehat{\theta Z} = \mu Tr\{1-[1-(2/\pi)\}\cos 2\theta-(4/\pi)(\theta\sin 2\theta-\cos 2\theta\ln r/c_0)]. \quad (51b)$$

On the axis of symmetry OP_0 we find

$$\sigma_2(r,0) = (2/\pi)\mu Tr(1+2\ln r/c_0). \qquad (52)$$

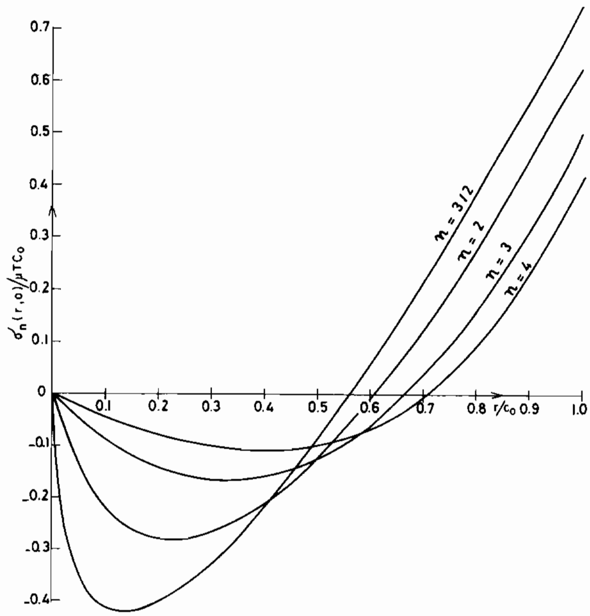

Fig. 11. Variation of shearing stress along the axes of symmetry of sectorial sections.

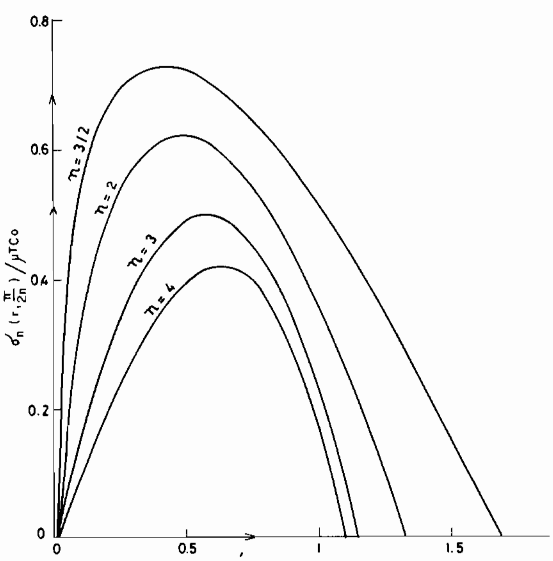

Fig. 12. Distribution of peripheral shearing stress along the radial edges of sectorial sections.

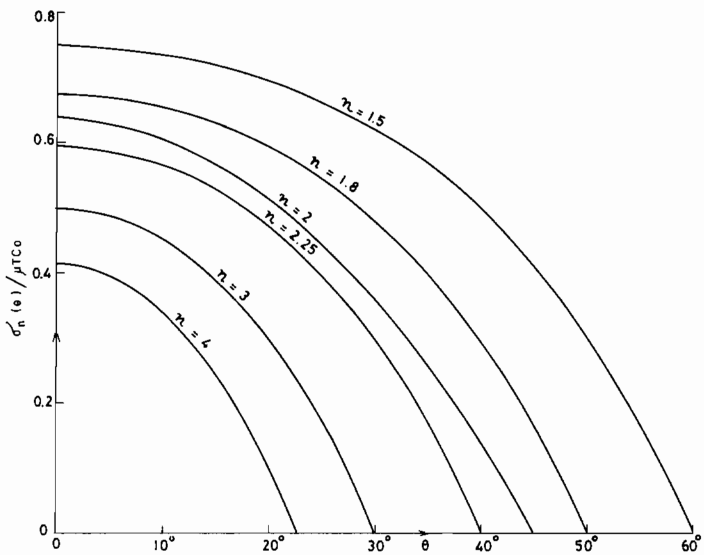

Fig. 13. Distribution of peripheral shearing stress along the curved bases of sectorial sections.

At any point $(r,\pi/4)$ of OP_0 we have

$$\sigma_2(r,\pi/4) = (\mu Tr/\pi)(2-\pi+4\ln r/c_0), \tag{53}$$

and at any point (r,θ) of P_0P_2 we get

$$\sigma_2(\theta) = (2/\pi)\,\mu Tr\{\cos^2 2\theta+\{\sin 2\theta+\theta\cos 2\theta+[2\theta\sin 2\theta-(\pi/2)]\tan 2\theta\}^2\}^{1/2}, \tag{54}$$

where r is given by (35).

The foregoing formulae are represented graphically in Figs 11, 12 and 13.

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حول مسألة اللى التقليدية لمقاطع معينة محدودة بمنحنيات أو على شكل قطاعات

وديع عطا الله بسالى وسميح عامر عبيد
قسم الرياضيات بجامعة الكويت

خلاصة

هذا البحث امتداد لبحثنا السابق الذى تناول ايجاد حلول لمسألة سانت فنانت الخاصة بليّ مقاطع معينة بسيطة الارتباط . والمقاطع المتماثلة التى عولجت هنا اما مقاطع منحنية محدودة بأقواس من قطاعات زائدة ذات اختلاف مركزي معين أو مقاطع على شكل قطاعات محدودة بخطين مستقيمين وقاعدات منحنية خاصة ، وقد أعطيت صيغ مضبوطة ومنتهية لدوال الاجهاد وصلابات اللىّ والقوى القاصة وعرضت النتائج العددية على شكل جداول ومنحنيات .