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Almost periodic solutions of a non-linear second order hyperbolic equation in a Hilbert space

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ABSTRACT

In this paper we prove existence and uniqueness of almost periodic solutions of a non-linear second order hyperbolic equation with constant operator coefficients in a Hilbert space.

INTRODUCTION

Let T_1, T_2, T_3, be bounded linear operators on a complex Hilbert space H and $F(x,y; v)$, $[(x,y) \in R^2, v \in H]$ be a mapping from $R^2 \times H$ to H. It was shown (Abdelaziz 1978) that under appropriate conditions on T_1, T_2, T_3 and F, the non-linear equation

$$\phi_{xy} + T_1\phi_x + T_2\phi_y + T_3\phi = F[x,y; \phi(x,y)] \qquad (*)$$

has a unique bounded solution $\phi : R^2 \to H$. The method used (Abdelaziz 1978) depends on reducing the differential equation (*) to a fixed point problem of the following operator

$$V\phi = \phi, \qquad \phi(.,.) \in C(R^2,H)$$

where V is the continuous mapping defined by

$$V\phi(x,y) = \int_K G(u,v; 0,0)\, F[x+u,y+v; \phi(x+u,y+v)]\, dudv.$$

In this paper we utilize this method to obtain existence and uniqueness of almost periodic solutions of (*).

First, we review a few facts and definitions concerning almost periodic functions. Let $f(.,.)$ be a continuous bounded function from R^2 to H, the supremum norm of $f(.,.)$ is defined by

$$\| f(.,.) \|_\infty = \sup \{ \| f(x,y) \| : (x,y) \in R^2 \}.$$

The set of all bounded continuous functions from R^2 to H is a Banach space under the sup-norm. This space is denoted by $C(R^2,H)$. A function $f \in C(R^2,H)$ is called (Bohr 1932) almost periodic if and only if the set of its translates is conditionally compact in $C(R^2,H)$, where a translate $f_{s,t}(.,.)$ of $f(.,.)$ is defined by

$$f_{s,t}(x,y) = f(x+s,y+t), (s,t) \in R^2.$$

This is also equivalent to the following: every sequence $\{f_{s_n,t_m}\}$ of translates of f has a convergent subsequence with respect to the sup-norm. We note that the set of almost periodic functions from the additive group of R^2 to H, forms a closed linear subspace of $C(R^2,H)$ which we denote by $AP(R^2,H)$. For further information about this subject see Dunford & Schwartz (1958).

EXISTENCE AND UNIQUENESS OF ALMOST PERIODIC SOLUTIONS

We shall make use of conditions I and II of Abdelaziz (1978) with II(ii) and II(iii) replaced by stronger forms. For convenience, we state these conditions in full.

T_1,T_2 and T_3 are linear bounded operators of H satisfying

I. (i) $T_1T_2 = T_2T_1$

(ii) The spectra of T_1 and T_2 are contained in the right half-plane $\{z : \operatorname{Re} z > 0\}$

(iii) The operator $T = T_3 - T_1T_2$ is positive self-adjoint.

F(x,y; v) is a mapping from $R^2 \times H$ to H such that:

II′. (i) F(x,y; v) is continuous in v for almost all (x,y)

(ii) F(x,y; v) is almost periodic in (x,y) for each fixed $v \in H$

(iii) $\exists$ a bounded function $\theta(.,.) : R^2 \rightarrow R^+$, such that

$$k \| \theta \|_\infty < [1-\exp(-c_1)]\,[1-\exp(-c_2)]$$

c_1,c_2 and k are as in (5) of Abdelaziz (1978)

(iv) F satisfies the following Lipschitz condition: for each pair $v,w \in H$

$$\| F(x,y;v) - F(x,y;w) \| \leq \theta(x,y) \, \| v - w \|, \text{ a.e.}$$

We now state and prove the main result of this paper.

Theorem. Let T_1,T_2 and T_3 satisfy conditions I and F(x,y;v) satisfy conditions II′. Then the differential equation (*) has a unique bounded solution which is almost periodic.

Proof. We will show that V maps $AP(R^2,H)$ into itself. This, together with the fact that V is a contraction mapping implies that V has a unique fixed point in $AP(R^2,H)$; and as in the proof of Theorem 8 (Abdelaziz 1978), this fixed point is the only bounded solution of (*).

Define a mapping W on $C(R^2,H)$ by

$$Wf(x,y) = \int_K G(u,v;0,0)\, f(x+u,y+v)\, du dv,$$

then W is a continuous linear operator of $C(R^2,H)$ into itself. Let $g = Wf$, $f \in AP(R^2,H)$. The set $\{g_{s,t}\}$,$(s,t) \in R^2$ is the image of the conditionally compact set $\{f_{s,t}\}$ under the continuous map W; hence it is also conditionally compact. Thus $g \in AP(R^2,H)$, i.e. W maps $AP(R^2,H)$ into itself. Now

$$V\phi(x,y) = W\{F[x,y;\phi(x,y)]\}$$

Thus we need only to show that $F[.,.;\phi(.,.)] \in AP(R^2,H)$ whenever $\phi(.,.) \in AP(R^2,H)$.

Assume that $\phi(.,.)$ is almost periodic, then $\phi(R^2)$ is conditionally compact and hence is totally bounded in H. Thus given $\varepsilon > 0$, $\exists$ $v_1, v_2,\ldots, v_n$ in H such that

$$\phi(R^2) \subseteq \bigcup_{i=1}^{n} N(v_i,\varepsilon),$$

where $N(v_i,\varepsilon)$ is the ε-ball with centre v_i. Put $G_i = \phi^{-1}[N(v_i,\varepsilon)]$, then

$$R^2 = \bigcup_i G_i.$$

Now for $(x,y) \in R^2$, $(x,y) \in G_i$ for some i, hence $\| \phi(x,y) - v_i \| < \varepsilon$, and we have

$$\begin{aligned} &\| F[s+x,t+y; \phi(x,y)] - F[x,y; \phi(x,y)] \| \leq \\ &\| F[s+x,t+y; \phi(x,y)] - F(s+x,t+y; v_i) \| + \\ &\| F(s+x,t+y; v_i) - F(x,y; v_i) \| + \\ &\| F(x,y; v_i) - F[x,y; \phi(x,y)] \|. \end{aligned}$$

Making use of Lemma 4 of Abdelaziz (1978), this is bounded by

$$\theta(s+x,t+y) \| \phi(x,y) - v_i \| + \theta(x,y) \| \phi(x,y) - v_i \| + \| F_{s,t}(x,y; v_i) - F(x,y; v_i) \|,$$

hence

$$\| F[s+x,t+y; \phi(x,y)] - F[x,y; \phi(x,y)] \| \leq 2\varepsilon \| \theta \|_\infty + \sum_{i=1}^{n} \| F_{s,t}(.,.; v_i) - F(.,.; v_i) \|_\infty \quad (1)$$

Also, we have

$$\| F[s+x,t+y; \phi(s+x,t+y)] - F(s+x,t+y; \phi(x,y)] \| \leq \| \theta \|_\infty \| \phi_{s,t}(.,.) - \phi(.,.) \|_\infty \quad (2)$$

combine (1) and (2), and recall that ε is arbitrary, we obtain

$$\| F_{s,t}[.,.; \phi(.,.)] - F[.,.; \phi(.,.)] \|_\infty \leq \sum_{i=1}^{n} \| F_{s,t}(.,.; v_i) - F(.,.; v_i) \|_\infty + \| \theta \|_\infty \| \phi_{s,t} - \phi \|_\infty \quad (3)$$

Note that by II′(ii), $F(.,.; v_i)$ is almost periodic. Now let $\{(s_n,t_n)\}$ be a sequence in R^2, choose a subsequence (call it) $\{(s_k,t_k)\}$ such that $\{\phi_{s_k,t_k}\}$ and $\{F_{s_k,t_k}(.,.; v_i)\}$, $1 \leq i \leq n$, are convergent in the sup-norm. Thus making use of (3) above, we see that $\{F_{s_k,t_k}[.,.; \phi(.,.)]\}$ is also convergent. This concludes the proof.

The above theorem applies in particular to systems of equations of the form

$$\frac{\partial^2 \phi_i}{\partial x \partial y} + \sum_{k=1}^{n} \left(a_{ik} \frac{\partial \phi_k}{\partial x} + b_{ik} \frac{\partial \phi_k}{\partial y} + c_{ik} \phi_k \right) = F_i[x,y; \phi_i(x,y)], \qquad i = 1, 2, \ldots, n.$$

where $\phi_i(x,y)$ is a real-valued function of two variables, and a_{ik}, b_{ik}, c_{ik} are scalars. In this case H is the n-dimensional Euclidean space, T_1, T_2 and T_3 are the $n \times n$ matrices (a_{ik}), (b_{ik}) and (c_{ik}) respectively. Solutions defined on a bounded rectangle of R^2 for such systems were also discussed in Germay (1928).

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حلول دورية تقريبا للمعادلة الهذلولية
من الرتبة الثانية وغير الخطية في فضاء هلبرت

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خلاصة

لقد أمكن في هذا البحث ، اثبات وجود ووحدانية حلول دورية تقريبا للمعادلات الهذلولية وغير الخطية من الرتبة الثانية ذات عوامل تأثير ثابتة في فضاء هلبرت .