J. Univ. Kuwait (Sci.) 2, 1975

L-CONVERGENCE OF CERTAIN COSINE SUMS

MAHER M. H. MARZUQ

Department of Mathematics, University of Kuwait

Abstract. The purpose of this paper is to obtain a generalization of a theorem of Garrett and Stanojevic (1975), concerning L-convergence of certain cosine sums.

1. A sequence $\{a_n\}$ is called convex if $\Delta^2 a_n \geq 0$, where $\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$ and $\Delta a_n = a_n - a_{n+1}$. It is said to be quasi-convex if

$$\Sigma_{k=1}^{\infty} (k+1) |\Delta^2 a_k| < \infty$$

In view of the well-known result that every bounded convex sequence $\{a_k\}$ satisfies the condition $\Sigma_{k=1}^{\infty} (k+1) \Delta^2 a_k < \infty$ (Bary 1964), it follows that such a sequence is necessarily quasi-convex. The converse is not true.

The concept of a null quasi-convex sequence has been further generalized by Telyakovskii (1973) in the following manner:

A sequence $\{a_k\}$ is said to satisfy condition S if

(1) $a_k = o(1)$, $k \to \infty$,

(2) numbers A_k exist such that $A_k \downarrow 0$ and $\Sigma_{k=1}^{\infty} A_k < \infty$,

(3) $|\Delta a_k| \leq A_k$ for all k

By taking $A_k = \Sigma_{m=k} |\Delta^2 a_m|$, we observe that every null quasi-convex sequence $\{a_k\}$ belongs to class S. As regards converse, it is clear from the example $a_k = \frac{(-1)^k}{k^2}$, that a sequence $\{a_k\} \in S$ need not necessarily be quasi-convex.

2. Let $g(x) = 1/_2\ a_o + \Sigma_{k=1}^{\infty} a_k \cos kx$, and

$$g_n(x) = 1/_2\ \Sigma_{k=o}^{n} \Delta a_k + \Sigma_{k=1}^{n} (\Sigma_{j=k}^{n} \Delta a_j) \cos kx.$$

Concerning the convergence of $g_n(x)$ in L-norm, Garrett and Stanojevic (1975), proved the following:

Theorem

If $\{a_n\}$ is a null quasi-convex sequence, then $g_n(x)$ converges to $g(x)$ in L-norm.

The object of this note is to generalize this theorem by using a less restrictive condition. We prove the following:

Theorem

If $\{a_n\} \in S$, then $g_n(x)$ converges to $g(x)$ in L-norm.

3. We need the following lemma (Sidon 1939), for the proof of our theorem:

Assume that the numbers α_i, $i=0, 1, \ldots, k$, satisfy the condition $|\alpha_i| \leq 1$. Then the following estimate is valid:

$$\int_o^{\pi} \left| \Sigma_{i=o}^{k} \alpha_i \frac{\sin(i+\frac{1}{2})x}{\sin x/2} \right| dx \leq C(k+1),$$

where C is an absolute constant.

4. *Proof of the theorem*:

Let $D_n(x) = 1/_2 + \cos x + \cos 2x + \ldots + \cos nx$

$$= \frac{\sin(n+1/_2)x}{2 \sin x/2}.$$

Using summation by parts we have

$$g(x) = \lim_{n\to\infty} [1/_2\ a_o + \Sigma_{k=1}^{n} a_k \cos kx]$$
$$= \lim_{n\to\infty} [1/2\ a_o + \Sigma_{k=1}^{n-1} D_k(x) \Delta a_k + a_n D_n(x) - 1/2\ a_o]$$
$$= \lim_{n\to\infty} [\Sigma_{k=o}^{n-1} D_k(x) \Delta a_k + a_n D_n(x)]$$
$$= \Sigma_{k=o}^{\infty} D_k(x) \Delta a_k, \text{ since } \lim_{n\to\infty} a_n D_n(x) = o \text{ if } x \neq O.$$

Similarly, summation by parts gives

$$g(x) = 1/2\ \Sigma_{k=o}^{n} \Delta a_k + \Sigma_{k=1}^{n} (\Sigma_{j=k}^{n} \Delta a_j) \cos kx$$
$$= \Sigma_{k=o}^{n} D_k(x) \Delta a_k.$$

Again using summation by parts we have

$$\Sigma_{n+1}^{m} D_k(x) \triangle a_k = \Sigma_{n+1}^{m} A_k D_k(x) \triangle \frac{a_k}{A_k}$$

$$= \Sigma_{n+1}^{m-1} T_k(x) \triangle A_k + A_m T_m(x) - A_{n+1} T_n(x),$$

where $T_n(x) = \Sigma_{k=1}^{n} D_k(x) \triangle \frac{a_k}{A_k}$

Taking $\alpha_k = \triangle \frac{a_k}{A_k}$, we observe by virtue of the above Lemma that

$$\int_0^\pi \left|\Sigma_{n+1}^{m} D_k(x) \triangle a_k\right| dx \leqslant \int_0^\pi \left|\Sigma_{n+1}^{m-1} T_k(x) \triangle A_k\right| dx + A_m \int_0^\pi |T_m(x)|\, dx + A_{n+1} \int_0^\pi |T_n(x)|\, dx \leqslant$$

$$\Sigma_{n+1}^{m-1} \triangle A_k \int_0^\pi |T_k(x)|\, dx + A_m \int_0^\pi |T_m(x)|\, dx + A_{n+1} \int_0^\pi |T_n(x)|\, dx \leqslant C \Sigma_{n+1}^{m-1} (k+1) \triangle A_k + C(m+1) A_m + C(n+1) A_{n+1}.$$

Making $m \to \infty$, we have

$$\int_0^\pi \left|\Sigma_{n+1}^{\infty} D_k(x) \triangle a_k\right| dx \leqslant C \Sigma_{n+1}^{\infty} (k+1) \triangle A_k + C(n+1) A_{n+1}$$

since $mA_m \to 0$, as $m \to \infty$

Hence we have

$$\int_0^\pi |g(x) - g_n(x)|\, dx = \int_0^\pi \left|\Sigma_{n+1}^{\infty} D_k(x) \triangle a_k\right| dx \leqslant C \Sigma_{n+1}^{\infty} (k+1) \triangle A_k + C(n+1) A_{n+1},$$

and, therefore,

$$\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx \leqslant C \lim_{n\to\infty} \Sigma_{n+1}^{\infty} (k+1) \triangle A_k + C \lim_{n\to\infty} (n+1) A_{n+1} = o(1),$$

since $\Sigma_{k=1}^{\infty} (k+1) \triangle A_k < \infty$. This proves the theorem.

5. Let $S_n(x)$ denote the n–th partial sum of $1/2\, a_0 + \Sigma_{k=1}^{\infty} a_k \cos kx$. Then it is easy to see that $g_n(x) = S_n(x) - a_{n+1} D_n(x)$.

Since $\int_0^\pi |D_n(x)|\, dx \sim \frac{2}{\pi} \log n$, we deduce the following corollary which is Theorem 4 of Telyakovskii (1973).

Corollary:

If $a_n \log n \to 0$, $n \to \infty$, and $\{a_n\} \in S$, then $\{S_n(x)\}$ converges in L–norm to $g(x)$ iff $a_n \log n \to 0$ as $n \to \infty$

I wish to express my deep gratitude to Professor S. M. Mazhar for his valuable guidance.

REFERENCES

Bary, N. 1964. A Treatise on Trigonometric series, vol. 1. Pergamon Press.

Garrett, J.W., & Stanojevic, C.V. 1975. On integrability and L-convergence of certain cosine sums. Notices Amer. Math. Soc. 22, A-166.

Sidon, S. 1939. Hinreichende Bedingungen für den Fourier-Charakter einer trigonometrischen Reihe. J. London Math. Soc. 14 : 158-160.

elyakovskii, S.A. 1973. Concerning a sufficient condition of Sidon for the integrability of trigonometric series. Math. Notes 14 : 742-748.

(*Received* 5 *June* 1975)

تقارب ــ ل لمجاميع جيوب التمام

ماهر محمد هاشم مرزوق

قسم الرياضيات بجامعة الكويت

خلاصـــــــة

الهدف من هذا البحث الحصول على تعميم لنظرية جاريت وستانوفيك (١٩٧٥) ، والتي تتعلق بتقارب ــ ل لمجاميع جيوب التمام .