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ON SOME CONFIGURATIONAL PROPOSITIONS

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Abstract. The purpose of this paper is to study the relationships among the configurational propositions of Fano, Desargues and Pappus. It is found that the proposition of Fano implies the second minor proposition of Pappus and that the proposition of Desargues implies three different minor forms of Pappus. A necessary and sufficient condition for a finite Desarguesian plane to be of an even order is found and an algebraic characterization of the satisfaction of the proposition of Fano is obtained in such a plane.

In a projective plane Π (Bumcrot 1969, P 25), let L and L' be two distinct lines with two sets of three points $\{1,2,3\}$ and $\{1',2',3'\}$, incident with them, respectively. Let $[1\,2'] \cap [1'\,2] = 3''$, $[1\,3'] \cap [1'\,3] = 2''$ and $[2\,3'] \cap [2'\,3] = 1''$.

Now there are three special, but non-degenerate, forms of the proposition of Pappus which may be stated as follows:

(1) If $2,2',2''$ are collinear, then $1'',2'',3''$ are also collinear. This is called the first minor proposition of Pappus.

(2) If two of the points $1'',2'',3''$ are collinear with the point $L \cap L'$, so is the third point. We shall refer to this statement as the second minor proposition of Pappus.

(3) If two triangles of an incomplete Pappian chain (Al-Dhahir 1957) are perspective from a point, then the chain can be completed. We shall call this the third minor form of Pappus.

In (Skornyakov 1953, P 25), it is shown that the Fano proposition implies the first minor proposition of Pappus. In what follows, we relate the Fano proposition to the second minor proposition of Pappus directly.

Theorem 1

In any projective plane, the Fano proposition implies the second minor proposition of Pappus.

Proof:

Let the points $0,2'',3''$ be collinear and assume that $[1\,1'] \cap [2\,2'] = 0'$ and $[1\,1'] \cap [3\,3'] = 0''$.

Applying the Fano proposition to the complete quadrangles $1\,2\,2'\,1'$ and $1\,3\,3'\,1'$ we obtain the two collinear sets $(0\,0'\,3'')$ and $(0\,0''\,2'')$ respectively. Since $0' = [0\,3''] \cap [1\,1']$, $0'' = [0\,2'']$

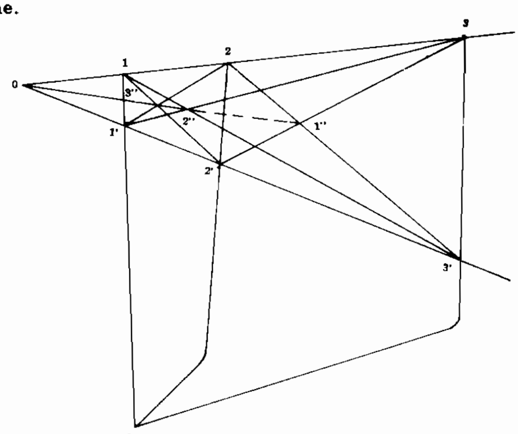

FIG. 1

$\cap [1\,1']$ and $(0\,2''\,3'')$, then $0' = 0''$. Now applying the Fano proposition to the complete quadrangle $2\,3\,3'\,2'$, we obtain the collinear set $(0\,0'\,1'')$ and hence the result.

We remark that the proposition of Fano trivially implies the third minor proposition of Pappus.

An interesting result in the theory of projective planes states that (Albert and Sandler 1968, P 78, P 82) the proposition of Pappus implies the theorem of Desargues, but the converse is false. Now, the question arises: what are the relationships between the proposition of Desargues and each of the preceding minor forms of Pappus ? In answering this question, we prove the following three theorems.

Theorem 2

In any projective plane, the proposition of Desargues implies the first minor proposition of Pappus.

Proof:

Let $[1\,2'] \cap [3\,1'] = 4$, $[1\,3'] \cap [3\,2'] = 5$ and $[4\,5] \cap [1'\,2'] = 6$

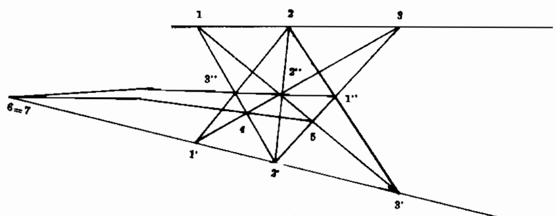

FIG. 2

Now, the two triangles 1' 2' 2 and 4 5 1 are perspective from 3 and hence, by the proposition of Desargues, they are perspective from the line (6 3'' 2''). Similarly, the two triangles 2'' 1' 3' and 2' 3'' 1'' are perspective from the point 2 and therefore they are perspective from the line (4 5 7) where $7 = [1'\,3'] \cap [1''\,3'']$. Since 6 is incident with [1' 3'] and [4 5] we have 6 = 7. Hence, (1'' 2'' 3'') is a collinear set.

Theorem 3

The second minor form of Pappus holds in every Desarguesian projective plane.

Proof:

Let $[1\,1'] \cap [2\,2'] = 0_3$, $[1\,2'] \cap [3\,1'] = 0_2$ and $[1\,3'] \cap [2\,1'] = 0_1$. Now, the two triangles 1 2'' 1' and 2 3'' 2' are perspective from 0 , hence $(0_1\,0_2\,0_3)$. Also the two triangles 1 3'' 1' and 3 2'' 3' are perspective from 0 and therefore $0_3\,I[3\,3']$. Since the two triangles 1' 2 3' and 1 2' 3 are perspective from 0_3 we have (0 1'' 3'') and the theorem follows.

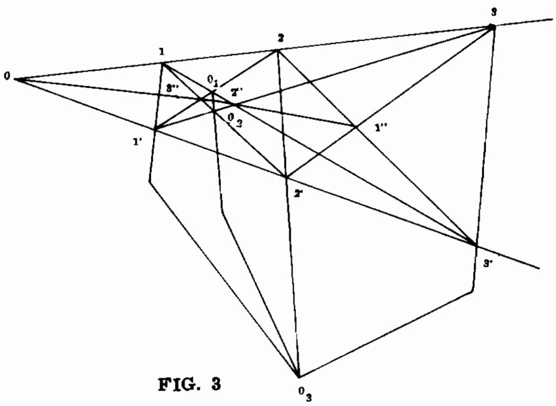

FIG. 3

Theorem 4

Let Π be a Desarguesian projective plane. Let T, T', T'' be three triangles in Π such that $T \supset T' \supset T'' \overset{\cdot}{\supset} T$ (Al-Dhahir 1957). If (T, T') are perspective from a point, then:

(1) $T \supset T' \supset T'' \supset T$; and

(2) (T' , T'') are perspective from a point, and (T, T'') are also perspective from a point

Proof:

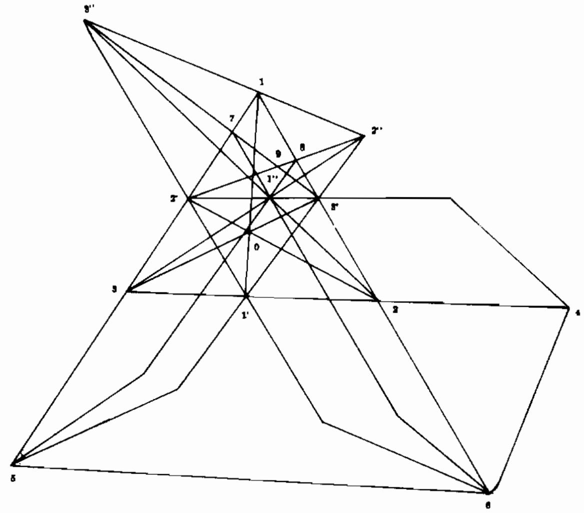

FIG. 4

Let $[5\,1''] \cap [1\,2] = 8$, $[6\,1''] \cap [1\,3] = 7$ and $[2'\,2''] \cap [3'\,3''] = 9$.

(1) Since the two triangles T : 1 2 3 and T' : 1' 2' 3' are perspective from 0, by the proposition of Desargues, we obtain (4 5 6). Also the two triangles 3' 1' 6 and 1'' 3 5 are perspective from 4 and therefore we have (2'' 8 2'). Similarly, since the two triangles 2' 1' 5 and 1'' 2 6 are perspective from 4, then we get (3'' 7 3'). Finally, the two triangles 2' 6 8 and 3' 7 5 are perspective from 1'' and hence (3'' 2'' 1).

(2) Since the two triangles 1 2'' 3' and 7 9 1'' are perspective from 2', then the three points 3'', 6, $[2''\,3'] \cap [9\,1'']$ are collinear. But $[2''\,3'] \cap [3''\,6] = 1'$, so we have (1' 1'' 9) and the two triangles T' : 1' 2' 3' and T'' : 1'' 2'' 3'' are perspective from 9. Similarly, one may prove that (T, T'') are also perspective. The proof is complete.

It is worthnoting that the second result given in theorem 4 above extends theorem 1 of (Al-Dhahir 1957) from Pappian to Desarguesian planes.

It has been shown (Bumcrot 1969, P 135) that if the Fano proposition is assumed in any finite projective plane Π , then Π is Desarguesian. Now, the following question presents itself: when does a finite Desarguesian plane satisfy the proposition of Fano ? An answer to this question is given in

Theorem 5

Let Π be a finite Desarguesian plane. Then the Fano proposition holds in Π if, and only if, Π is of even order.

Proof:

Let Π be a Desarguesian plane of order n; R its coordinatizing set. Then (R, +, .) is a field of order n (Albert & Sandler 1968, P 77).

A) Let the proposition of Fano hold. Then $1+1=0$ (Skornyakov 1953, P 24). Hence $(\{0,1\},+)$ is a subgroup of the group (R, +). Since the order of a subgroup of a group G must divide the order of the group, 2 must divide n and consequently n must be even.

B) Let n be even. Since (R, +, .) is a field, then (R, +) is an abelian group of even order and (R, +) has a subgroup of order 2. Hence there exists $0 \neq a \in R$ such that $a+a=0$, or $1+1=0$ and therefore the Fano proposition must hold. This completes the proof of the theorem.

We conclude the paper with the following consequence of theorem 5.

Theorem 6

In any finite Desarguesian plane Π of order n, the proposition of Fano holds if, and only if, $a.a \neq 1$ for all $a \neq 1$ in R.

Proof:

Π is Desarguesian implies that (R, +, .) is a field (Albert & Sandler 1968, P 77).

A) If the proposition of Fano holds then n is even, by theorem 5, and hence $(R-\{0\},.)$ is an abelian group of order $n-1$. Hence, there is no subgroup of order 2 of the group $(R-\{0\},.)$. In other words, $a.a \neq 1$ for all $a \neq 1$ in R.

B) Let $a.a \neq 1$ for all $a \neq 1$ in R. Then there exists no subgroup of order 2 of the group $(R-\{0\},.)$. This implies that $(R-\{0\},.)$ is of odd order. Hence n is even and, using theorem 5, the Fano proposition must follow.

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حول بعض القضايا التشكلية

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خلاصـــة

يتضمن هذا البحث دراسة للعلاقات بين قضايا فانو وديسارك وبابس التشكيلية . وامكن اثبات أن قضية فانو تقود الى قضية بابس الصغيرة الثانية وان قضية ديسارك تؤدي الى قضايا بابس الصغيرة المختلفة . كذلك أمكن التوصل الى الشرط الضروري والكافي لتكون رتبة مستوى ديسارك المنتهي عددا زوجيا ، وبذلك تم تعيين توافـر قضية فانو في هذا المستوى جبريا .