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On eigen-forms on the 2-sphere

ALOIS SVEC

Department of Mathematics, University of Kuwait

ABSTRACT

We prove that, on the unit 2-sphere, the only eigen-value of the Laplacian on 1-forms less than 4 is equal to 2. The corresponding eigen-forms are described completely.

INTRODUCTION

Let (M,g) be an orientable compact connected n-dimensional Riemannian manifold. To each p-form ω on M, there is associated the $(p+1)$-form $d\omega$ and the $(n-p)$-form $*\omega$ respectively, $*$ being the Hodge operator.

The exterior codifferential δ is then defined by

$$\delta\omega = (-1)^p *^{-1} d*\omega, \tag{1}$$

$*^{-1}$ being the inverse mapping to $*$; see Berger *et al.* (1971) and De Rham (1960). Finally, the Laplacian Δ on p-forms is given by

$$\Delta\omega = (d\delta + \delta d)\omega. \tag{2}$$

We say that $\lambda \in \mathbf{R}$ belongs to $\mathrm{Spec}^{(p)}(\Delta)$ if there is a non-trivial p-form ω on M such that

$$\Delta\omega = \lambda\omega. \tag{3}$$

The general problem is to exhibit $\mathrm{Spec}^{(p)}(\Delta)$ for a given (M,g). Up to now, little is known. $\mathrm{Spec}^{(0)}(\Delta)$ is known just for the hypersphere and several simple homogeneous manifolds. On the other hand, for $n = 2$, $\mathrm{Spec}^{(0)}(\Delta)$ does not contain the intervals

$$(-\infty,0),\ (0,2\kappa_0),\ (2\kappa_1,6\kappa_0) \tag{4}$$

with

$$\kappa_0 = \min_M K, \qquad \kappa_1 = \max_M K, \tag{5}$$

K being the Gauss curvature of (M,g). Furthermore, there are no general methods. In what follows, I am going to use just the Stokes theorem

$$\int_M d\phi = 0, \tag{6}$$

ϕ being an $(n-1)$-form.

LINEAR FORMS ON THE UNIT 2-SPHERE

The purpose of this note is to prove the following:

Theorem. Let (M,g) be the unit sphere of the Euclidean space E^3, g being the induced metric. Let $\lambda < 4$ and $\lambda \in \mathrm{Spec}^{(1)}(\Delta)$. Then $\lambda = 2$. In E^3, take two vectors **v**, **w**. The 1-forms $\omega_{\mathbf{v}},\omega_{\mathbf{w}}$ on M are defined by

$$\omega_{\mathbf{v}}(t) = \langle \mathbf{v},t\rangle,\ \omega_{\mathbf{w}}(t) = \langle \mathbf{w},t\rangle;\ t \in T(M); \tag{7}$$

$\langle\ ,\ \rangle$ being the scalar product in E^3. Then the 1-form

$$\omega = \omega_{\mathbf{v}} + *\omega_{\mathbf{w}} \tag{8}$$

is the most general eigen-form satisfying $\Delta\omega = 2\omega$.

Proof. Let $M \subset E^3$ be the unit sphere; we are going to investigate its coordinate neighbourhood $U \subset M$. To each point $m \in M$, let us associate an orthonormal frame $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ such that $m + \mathbf{v}_3$ is the centre of our sphere. Then we have

$$dm = \omega^1\mathbf{v}_1 + \omega^2\mathbf{v}_2, \qquad d\mathbf{v}_1 = \omega_1^2\mathbf{v}_2 + \omega^1\mathbf{v}_3,$$
$$d\mathbf{v}_2 = -\omega_1^2\mathbf{v}_1 + \omega^2\mathbf{v}_3, \qquad d\mathbf{v}_3 = -\omega^1\mathbf{v}_1 - \omega^2\mathbf{v}_2, \tag{9}$$

the 1-forms ω^1, ω^2, ω_1^2 satisfying

$$d\omega^1 = -\omega^2 \wedge \omega_1^2, \qquad d\omega^2 = \omega^1 \wedge \omega_1^2, \qquad d\omega_1^2 = -\omega^1 \wedge \omega^2. \tag{10}$$

On M, be given a 1-form ω; in U, we may write

$$\omega = a\omega^1 + b\omega^2. \tag{11}$$

$a,b\colon U \to \mathbf{R}$ being functions. The functions $a_i,b_i\colon U \to \mathbf{R}$ are defined by

$$da - b\omega_1^2 = a_1\omega^1 + a_2\omega^2,$$
$$db + a\omega_1^2 = b_1\omega^1 + b_2\omega^2. \tag{12}$$

The exterior differentiation implies

$$\{da_1 - (a_2+b_1)\omega_1^2\} \wedge \omega^1 + \{da_2 + (a_1-b_2)\omega_1^2\} \wedge \omega^2 = b\omega^1 \wedge \omega^2,$$
$$\{db_1 + (a_1-b_2)\omega_1^2\} \wedge \omega^1 + \{db_2 + (a_2+b_1)\omega_1^2\} \wedge \omega^2 = -a\omega^1 \wedge \omega^2 \tag{13}$$

and the existence of functions a_{ij}, $b_{ij}\colon U \to \mathbf{R}$ such that

$$da_1 - (a_2+b_1)\omega_1^2 = a_{11}\omega^1 + (a_{12} - \tfrac{1}{2}b)\omega^2,$$
$$da_2 + (a_1-b_2)\omega_1^2 = (a_{12} + \tfrac{1}{2}b)\omega^1 + a_{22}\omega^2,$$
$$db_1 + (a_1-b_2)\omega_1^2 = b_{11}\omega^1 + (b_{12} + \tfrac{1}{2}a)\omega^2,$$
$$db_2 + (a_2+b_1)\omega_1^2 = (b_{12} - \tfrac{1}{2}a)\omega^1 + b_{22}\omega^2. \tag{14}$$

The differential consequences of (14) being

$$\{da_{11} - (2a_{12}+b_{11})\omega_1^2\} \wedge \omega^1 + \{da_{12} + (a_{11}-a_{22}-b_{12})\omega_1^2\} \wedge \omega^2 = (a_2 + \tfrac{3}{2}b_1)\omega^1 \wedge \omega^2,$$
$$\{da_{12} + (a_{11}-a_{22}-b_{12})\omega_1^2\} \wedge \omega^1 + \{da_{22} + (2a_{12}-b_{22})\omega_1^2\} \wedge \omega^2 = (\tfrac{3}{2}b_2 - a_1)\omega^1 \wedge \omega^2,$$
$$\{db_{11} + (a_{11}-2b_{12})\omega_1^2\} \wedge \omega^1 + \{db_{12} + (a_{12}+b_{11}-b_{22})\omega_1^2\} \wedge \omega^2 = (b_2 - \tfrac{3}{2}a_1)\omega^1 \wedge \omega^2,$$
$$\{db_{12} + (a_{12}+b_{11}-b_{22})\omega_1^2\} \wedge \omega^1 + \{db_{22} + (a_{22}+2b_{12})\omega_1^2\} \wedge \omega^2 = -(\tfrac{3}{2}a_2 + b_1)\omega^1 \wedge \omega^2. \tag{15}$$

we get the existence of functions A_α, $B_\alpha: U \to \mathbf{R}$ satisfying

$$\begin{aligned}
da_{11}-(2a_{12}+b_{11})\omega_1^2 &= A_1\omega^1+(A_2-\tfrac{1}{2}a_2-\tfrac{3}{4}b_1)\omega^2,\\
da_{12}+(a_{11}-a_{22}-b_{12})\omega_1^2 &= (A_2+\tfrac{1}{2}a_2+\tfrac{3}{4}b_1)\omega^1+(A_3+\tfrac{1}{2}a_1-\tfrac{3}{4}b_2)\omega^2,\\
da_{22}+(2a_{12}-b_{22})\omega_1^2 &= (A_3-\tfrac{1}{2}a_1+\tfrac{3}{4}b_2)\omega^1+A_4\omega^2,\\
db_{11}+(a_{11}-2b_{12})\omega_1^2 &= B_1\omega^1+(B_2+\tfrac{3}{4}a_1-\tfrac{1}{2}b_2)\omega^2,\\
db_{12}+(a_{12}+b_{11}-b_{22})\omega_1^2 &= (B_2-\tfrac{3}{4}a_1+\tfrac{1}{2}b_2)\omega^1+(B_3+\tfrac{3}{4}a_2+\tfrac{1}{2}b_1)\omega^2,\\
db_{22}+(a_{22}+2b_{12})\omega_1^2 &= (B_3-\tfrac{3}{4}a_2-\tfrac{1}{2}b_1)\omega^1+B_4\omega^2. \qquad (16)
\end{aligned}$$

For 1-forms, we have

$$*(p\omega^1+q\omega^2) = -q\omega^1+p\omega^2, \qquad *^{-1}(p'\omega^1+q'\omega^2) = q'\omega^1-p'\omega^2; \qquad (17)$$

$$\Delta\omega = (*^{-1}d*\mathrm{d}-d*^{-1}d*)\omega. \qquad (18)$$

In our case,

$$\begin{aligned}
d\omega = (b_1-a_2)\omega^1 \wedge \omega^2, &\qquad *d\omega = b_1-a_2,\\
d*\mathrm{d}\omega &= (b_{11}-a_{12}-\tfrac{1}{2}b)\omega^1+(b_{12}-a_{22}+\tfrac{1}{2}a)\omega^2,\\
*^{-1}d*d\omega &= (b_{12}-a_{22}+\tfrac{1}{2}a)\omega^1+(a_{12}-b_{11}+\tfrac{1}{2}b)\omega^2,\\
\omega = -b\omega^1+a\omega^2, \qquad d\omega &= (a_1+b_2)\omega^1 \wedge \omega^2, \qquad *^{-1}d*\omega = a_1+b_2,\\
d*^{-1}d*\omega &= (a_{11}+b_{12}-\tfrac{1}{2}a)\omega^1+(a_{12}+b_{22}-\tfrac{1}{2}b)\omega^2. \qquad (19)
\end{aligned}$$

Thus, for the 1-form ω (11),

$$\Delta\omega = (a-a_{11}-a_{22})\omega^1+(b-b_{11}-b_{22})\omega^2. \qquad (20)$$

The form ω satisfying (6), we get

$$a_{11}+a_{22} = (1-\lambda)a, \qquad b_{11}+b_{22} = (1-\lambda)b. \qquad (21)$$

Because of (16), we obtain

$$\begin{aligned}
A_1+A_3 = (\tfrac{3}{2}-\lambda)a_1-\tfrac{3}{4}b_2, &\qquad A_2+A_4 = (\tfrac{3}{2}-\lambda)a_2+\tfrac{3}{4}b_1,\\
B_1+B_3 = \tfrac{3}{4}a_2+(\tfrac{3}{2}-\lambda)b_1, &\qquad B_2+B_4 = -\tfrac{3}{4}a_1+(\tfrac{3}{2}-\lambda)b_2. \qquad (22)
\end{aligned}$$

For a general 1-form ω (11), we easily get

$$\begin{aligned}
&d*d\{(a_1-b_2)^2+(a_2+b_1)^2\} =\\
&= [2\{(a_{11}-b_{12}+\tfrac{1}{2}a)^2+(a_{12}+b_{11}+\tfrac{1}{2}b)^2+(a_{12}-b_{22}-\tfrac{1}{2}b)^2+(a_{22}+b_{12}+\tfrac{1}{2}a)^2\}+\\
&\qquad +\tfrac{7}{2}\{(a_1-b_2)^2+(a_2+b_1)^2\}+\\
&\qquad +2\{(a_1-b_2)(A_1+A_3-B_2-B_4)+(a_2+b_1)(A_2+A_4+B_1+B_3)\}]\omega^1 \wedge \omega^2. \qquad (23)
\end{aligned}$$

For an eigen-form, we have (22) and

$$\begin{aligned}
&d*d\{(a_1-b_2)^2+(a_2+b_1)^2\} =\\
&= [2\{(a_{11}-b_{12}+\tfrac{1}{2}a)^2+(a_{12}+b_{11}+\tfrac{1}{2}b)^2+(a_{12}-b_{22}-\tfrac{1}{2}b)^2+(a_{22}+b_{12}+\tfrac{1}{2}a)^2\}+\\
&\qquad +2(4-(a_1-b_2)^2+(a_2+b_1)^2\}]\omega^1 \wedge \omega^2. \qquad (24)
\end{aligned}$$

Using the Stokes theorem and the supposition $4-\lambda > 0$, we get

$$a_1-b_2 = 0, \qquad a_2+b_1 = 0, \tag{25}$$

$$a_{11}-b_{12}+\tfrac{1}{2}a = a_{12}+b_{11}+\tfrac{1}{2}b = a_{12}-b_{22}-\tfrac{1}{2}b = a_{22}+b_{12}+\tfrac{1}{2}a = 0; \tag{26}$$

It is easy to see that (26) is just a differential consequence of (25).
Let us introduce the functions $r,s: U \to \mathbf{R}$ by

$$r = a_1 = b_2, \qquad s = -a_2 = b_1; \tag{27}$$

the equation (12) is then

$$\begin{aligned} da-b\omega_1^2 &= r\omega^1-s\omega^2, \\ db+a\omega_1^2 &= s\omega^1+r\omega^2. \end{aligned} \tag{28}$$

Using successive differentiations and the Cartan's lemma, we get the existence of functions $x,y,t,z,p,q: U \to \mathbf{R}$ such that

$$\begin{aligned} dr \wedge \omega^1-ds \wedge \omega^2 &= b\omega^1 \wedge \omega^2, \\ ds \wedge \omega^1+dr \wedge \omega^2 &= -a\omega^1 \wedge \omega^2, \end{aligned} \tag{29}$$

$$\begin{aligned} dr &= (x-\tfrac{1}{2}a)\omega^1-(y+\tfrac{1}{2}b)\omega^2, \\ ds &= (y-\tfrac{1}{2}b)\omega^1+(x+\tfrac{1}{2}a)\omega^2; \end{aligned} \tag{30}$$

$$\begin{aligned} (dx+y\omega_1^2) \wedge \omega^1-(dy-x\omega_1^2) \wedge \omega^2 &= s\omega^1 \wedge \omega^2, \\ (dy-x\omega_1^2) \wedge \omega^1+(dx+y\omega_1^2) \wedge \omega^2 &= -r\omega^1 \wedge \omega^2, \end{aligned} \tag{31}$$

$$\begin{aligned} dx+y\omega_1^2 &= (t-\tfrac{1}{2}r)\omega^1-(z+\tfrac{1}{2}s)\omega^2, \\ dy-x\omega_1^2 &= (z-\tfrac{1}{2}s)\omega^1+(t+\tfrac{1}{2}r)\omega^2; \end{aligned} \tag{32}$$

$$\begin{aligned} (dt+2z\omega_1^2) \wedge \omega^1-(dz-2t\omega_1^2) \wedge \omega^2 &= 0, \\ (dz-2t\omega_1^2) \wedge \omega^1+(dt+2z\omega_1^2) \wedge \omega^2 &= 0, \end{aligned} \tag{33}$$

$$\begin{aligned} dt+2z\omega_1^2 &= p\omega^1-q\omega^2, \\ dz-2t\omega_1^2 &= q\omega^1+p\omega^2. \end{aligned} \tag{34}$$

Now,

$$d*d\{(x-\tfrac{1}{2}a)^2+(y+\tfrac{1}{2}b)^2+r^2\} = 4(z^2+t^2)\omega^1 \wedge \omega^2, \tag{35}$$

and the Stokes theorem implies

$$z = t = 0. \tag{36}$$

Let us consider the vectors

$$\begin{aligned} \mathbf{v} &= (\tfrac{1}{2}a-x)\mathbf{v}_1+(y+\tfrac{1}{2}b)\mathbf{v}_2+r\mathbf{v}_3, \\ \mathbf{w} &= (\tfrac{1}{2}b-y)\mathbf{v}_1-(x+\tfrac{1}{2}a)\mathbf{v}_2+s\mathbf{v}_3; \end{aligned} \tag{37}$$

It is easy to see that

$$d\mathbf{v} = d\mathbf{w} = 0, \tag{38}$$

i.e., $\mathbf{v}$ and $\mathbf{w}$ are fixed vectors of E^3. The forms (7) are then

$$\omega_{\mathbf{v}} = (\tfrac{1}{2}a-x)\omega^1+(y+\tfrac{1}{2}b)\omega_1^2, \qquad \omega_{\mathbf{w}} = (\tfrac{1}{2}b-y)\omega^1-(x+\tfrac{1}{2}a)\omega^2, \tag{39}$$

and we get exactly (8).

Finally, let

$$\mathbf{v} = e\mathbf{v}_1+f\mathbf{v}_2+g\mathbf{v}_3, \qquad \mathbf{w} = k\mathbf{v}_1+l\mathbf{v}_2+m\mathbf{v}_3 \tag{40}$$

be fixed vectors in E^3. From

$$0 = d\mathbf{v} = (de-f\omega_1^2-g\omega^1)\mathbf{v}_1+(df+e\omega_1^2-g\omega^2)\mathbf{v}_2+(dg+e\omega^1+f\omega^2)\mathbf{v}_3, \tag{41}$$

we get

$$de-f\omega_1^2 = g\omega^1, \qquad df+e\omega_1^2 = g\omega^2, \qquad dg = -e\omega^1-f\omega^2 \tag{42}$$

and similar equations

$$dk-l\omega_1^2 = m\omega^1, \qquad dl+k\omega_1^2 = m\omega^2, \qquad dm = -k\omega^1-l\omega^2 \tag{43}$$

for $\mathbf{w}$. We have

$$\omega_{\mathbf{v}} = e\omega^1+f\omega^2, \qquad \omega_{\mathbf{w}} = k\omega^1+l\omega^2, \tag{44}$$

i.e.

$$\omega = \omega_{\mathbf{v}}+*\omega_{\mathbf{w}} = (e+l)\omega^1+(f+k)\omega^2; \tag{45}$$

$$d\omega = 2m\omega^1\wedge\omega^2, \qquad *d\omega = 2m, \qquad d*\mathrm{d}\omega = -2k\omega^1-2l\omega^2,$$

$$*^{-1}d*d\omega = -2l\omega^1+2k\omega^2;$$

$$*\omega = -(f+k)\omega^1+(e-l)\omega^2, \qquad d*\omega = 2g\omega^1\wedge\omega^2, \qquad *^{-1}d*\omega = 2g,$$

$$d*^{-1}d*\omega = -2e\omega^1-2f\omega^2, \tag{46}$$

i.e.

$$\Delta\omega = 2(e-l)\omega^1+2(f+k)\omega^2 = 2\omega. \tag{47}$$

This proves our theorem.

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حول الصيغ الذاتية على الكرة ـ ٢

الويس شفيك
قسم الرياضيات بجامعة الكويت

خلاصة

لقد أثبت المؤلف بأنه ، على الكرة ـ ٢ ذات نصف قطر يساوي الوحدة ، تكون القيمة الذاتية الوحيدة للابلاسي في الصيغة ـ ١ أقل من أربعة تساوي ٢. كما أمكن تحديد الصيغ المناظرة بالكامل .