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FREE VIBRATION OF PARALLELOGRAM PLATES WITH CLAMPED EDGES

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Abstract. An approximate method for the determination of natural frequencies of clamped elastic skew plates is presented. The classical differential equation of motion for the transverse displacement of a plate is assumed to apply. The deflection is expressed in terms of an infinite series. The boundary conditions of zero slope and deflection are satisfied exactly. A modified Galerkin technique is used to minimize the error resulting from the substitution of the proposed deflection function into the equilibrium equation.

Introduction

The vibration problem of clamped rectangular plates has received a great deal of attention, especially for the case of a square plate. The first reasonably accurate results for a square plate were given by Sezawa (1931). A summary of different approximate and exact techniques available in the literature was given by Leissa (1969).

For a parallelogram plate, no exact solution to the equation of motion in skew coordinates is known to exist in a form with separable variables due to the complexity of the differential equation. However, some solutions have been obtained by approximate methods. Kaul and Cadambe (1956) used the Rayleigh - Ritz method and the product of characteristic beam function to obtain approximate upper bounds for the frequency parameters. They computed the lower bounds by the use of the Kato - Temple method. Conway and Farnham (1965) analysed the case of a rhombus plate by the use of the point - matching method. Further approximate results were obtained by Hasegawa (1957) who used the Rayleigh - Ritz method and by Hamada and Konda (1957) and Hamada (1959) who used the method of Trefftz. Experimental results for the rhombic plate were also given by Hamada and Konda (1957) and Hamada (1959).

This study deals with the transverse vibration of clamped skew elastic plates. The deflection function is represented by an infinite series which satisfies term by term the conditions of zero slope and deflection at the boundaries. Frequency parameters obtained for various values of angle of skewness α and aspect ratio γ agree very closely with other known results.

Analysis

The classical differential equation of motion for the transverse displacement $\overline{W}(x,y,t)$ of a flat plate is (Leissa 1969)

$$D \nabla^4 \overline{W} + \sigma \frac{\partial^2 \overline{W}}{\partial t^2} = 0 \qquad (1)$$

where D is the flexural rigidity and is defined by

$$D = \frac{E h^3}{12 (1 - \nu^2)} \qquad (2)$$

E is Young's modulus, h is the plate thickness, ν is Poisson's ratio, σ is mass density per unit area of the plate, t is time, and ∇^4 is the biharmonic operator.

The skew coordinates x, y of a point P are shown in Fig. 1. The skew coordinates are related to rectangular coordinates by

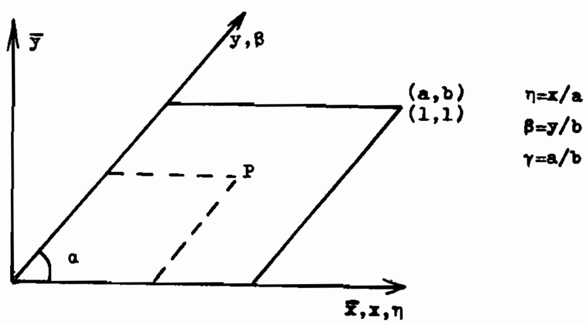

FIG. 1 - Coordinate systems.

$$x = \bar{x} - \bar{y} / \tan \alpha \qquad (3)$$

$$y = \bar{y} / \sin \alpha$$

The Laplacian operator in skew coordinates was given by Morley (1963) as

$$\nabla^2 = \frac{1}{\sin^2\alpha}\left(\frac{\partial^2}{\partial x^2} - 2\cos\alpha\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\right) \quad (4)$$

It is convenient to introduce the non-dimensional coordinates $\eta = x/a$, $\beta = y/b$ and the aspect ratio $\gamma = a/b$ as shown in Fig. 1. The Laplacian and biharmonic operators in non-dimensional coordinates are,

$$\nabla^2 = \frac{1}{a^2\sin^2\alpha}\left(\frac{\partial^2}{\partial\eta^2} - 2\gamma\cos\alpha\frac{\partial^2}{\partial\eta\,\partial\beta} + \gamma^2\frac{\partial^2}{\partial\beta^2}\right) \quad (5)$$

and

$$\nabla^4 = \frac{1}{a^4\sin^4\alpha}\left[\frac{\partial^4}{\partial\eta^4} - 4\gamma\cos\alpha\frac{\partial^4}{\partial\eta^3\partial\beta} + 2\gamma^2(1+2\cos^2\alpha)\frac{\partial^4}{\partial\eta^2\partial\beta^2} - 4\gamma^3\cos\alpha\frac{\partial^4}{\partial\eta\,\partial\beta^3} + \gamma^4\frac{\partial^4}{\partial\beta^4}\right] \quad (6)$$

respectively.

The boundary conditions that must be satisfied for clamped plates are

$$\begin{aligned} &\overline{W} = 0, \quad \overline{W}_\eta = 0 \quad \text{at } \eta = 0 \text{ and } \eta = 1 \\ &\overline{W} = 0, \quad \overline{W}_\beta = 0 \quad \text{at } \beta = 0 \text{ and } \beta = 1 \end{aligned} \quad (7)$$

When free vibrations are assumed, the motion is expressed as

$$\overline{W}(\eta,\beta,t) = W(\eta,\beta)\cos\varphi t \quad (8)$$

where φ is the circular frequency (expressed in radians / unit time) and W is a function of the position coordinates only. Substituting equation (8) into equation (1) yields

$$\nabla^4 W = \lambda^2 W$$

in which

$$\lambda^2 = \frac{\sigma a^4 \sin^4\alpha\ \varphi^2}{D} \quad (9)$$

The skew plate deflection function $W(\eta,\beta)$ is chosen as

$$W = \sum_{m=1}^{K}\sum_{n=1}^{K} T_{mn} F_{mn}(\eta,\beta) \quad (10)$$

in which K represents the order of approximation, T_{mn} are arbitrary constants and

$$F_{mn}(\eta,\beta) = \left[\eta(1-(2+(-1)^m)\eta + (1+(-1)^m)\eta^2) - \frac{1}{\mu_m}\sin\mu_m\eta\right]\left[\beta(1-(2+(-1)^n)\beta + (1+(-1)^n)\beta^2) - \frac{1}{\mu_n}\sin\mu_n\beta\right] \quad (11)$$

where $\mu_m = m\pi$, $\mu_n = n\pi$, $m,n = 1, 2, 3, \ldots, K$. This expression for W satisfies the boundary conditions given in equation (7) exactly term by term.

Let $E(\eta,\beta)$ be the error resulting from the substitution of equation (10) into (1), i.e.,

$$E(\eta,\beta) = (\nabla^4 - \lambda^2)\, W \quad (12)$$

This error is minimized by a modified Galerkin technique. If $R_{ij}(\eta,\beta)$ represent any set of orthogonal linearly independent functions, $E(\eta,\beta)$ can be expressed as

$$E(\eta,\beta) = \sum_{i=1}^{K}\sum_{j=1}^{K} B_{ij} R_{ij}(\eta,\beta) \quad (13)$$

The constants coefficients of this series are given by

$$B_{ij} = \frac{\int_0^1\int_0^1 E(\eta,\beta) R_{ij}(\eta,\beta)\, d\eta\, d\beta}{\int_0^1\int_0^1 R_{ij}^2(\eta,\beta)\, d\eta\, d\beta} \quad (14)$$

where $i,j = 1,2,3, \ldots, K$. Each of the coefficients of the series are now set equal to zero. This results into the following K^2 equations

$$\int_0^1\int_0^1 E(\eta,\beta) R_{ij}(\eta,\beta)\, d\eta\, d\beta = 0 \quad (15)$$

In this study, the functions $R_{ij}(\eta,\beta)$ are taken as $\sin\mu_i\eta \sin\mu_j\beta$. Upon substituting these values of $R_{ij}(\eta,\beta)$ into equation (15) and making use of equation (12) one gets

$$\int_0^1\int_0^1 [(\nabla^4 - \lambda^2) W] \sin\mu_i\eta \sin\mu_j\beta\, d\eta\, d\beta = 0 \text{ for } i,j = 1,2,3, \ldots, K \quad (16)$$

The substitution for W from equation (10) into (16) yields

$$\sum_{m=1}^{K}\sum_{n=1}^{K} T_{mn}\int_0^1\int_0^1 [(\nabla^4 - \lambda^2) F_{mn}(\eta,\beta)] \sin\mu_i\eta \sin\mu_j\beta\, d\eta\, d\beta = 0 \quad (17)$$

Equation (17) can be written in a matrix form as

$$[A - \lambda^2 B]\{T\} = 0 \quad (18)$$

in which [A] and [B] are coefficients matrices defined by

$$A_{pq} = \int_0^1 \int_0^1 (\nabla^4 F_{mn}(\eta,\beta)) \sin\mu_i \eta \sin\mu_j \beta \, d\eta \, d\beta$$

$$B_{pq} = \int_0^1 \int_0^1 F_{mn}(\eta,\beta) \sin\mu_i \eta \sin\mu_j \beta \, d\eta \, d\beta$$

$$\{T\} = \{T_q\}$$

where $p = i + (j-1)K, \quad i,j = 1,2,3,\ldots,K$

$$q = m + (n-1)K, \quad m,n = 1,2,3,\ldots,K$$

In order to evaluate the values of λ^2 from equation (18), the inverse of the matrix $[B]$ is first found. Thus one gets

$$[C - \lambda^2 I]\{T\} = 0$$

where I is the identity matrix and [C] is a square matrix of order $(K^2 \times K^2)$ defined by

$$[C] = [B]^{-1}[A]$$

To obtain a non-trivial solution for equations (19), the determinant of the coefficients matrix must vanish, i.e.

$$|[C - \lambda^2 I]| = 0 \tag{20}$$

Since the matrix is of the order $(K^2 \times K^2)$, we obtain K^2 values for λ^2. The corresponding K^2 natural oscillation frequencies of the plate can now be computed from

$$\varphi = \frac{\lambda}{a^2 \sin^2 \alpha}\sqrt{D/\sigma} \tag{21}$$

Numerical Results

The values of λ^2 for various values of γ and α were computed by programming the proposed analysis on an electronic computer. These results are presented in Tables 1-4. To evaluate the accuracy and convergence of the method, Table 1 gives the lowest ten frequency parameters for a plate having $\alpha = 60^0$ and $\gamma = 1.0$ for three values of K.

TABLE 1. The lowest ten values of λ for various values of K. $\gamma = 1.0$ $\alpha = 60^0$, $\lambda = \varphi\sqrt{\sigma/D}\ a^2 \sin^2\alpha$

λ \ K	5	6	7
1	34.519	34.573	34.550
2	61.507	61.207	61.280
3	78.993	78.889	78.869
4	90.151	89.605	89.593
5	124.50	124.69	123.85
6	124.82	124.91	124.16
7	139.91	140.09	139.63
8	170.48	164.80	164.72
9	170.63	164.90	164.92
10	209.25	200.23	200.58

TABLE 2. Frequency Parameter $\varphi a^2\sqrt{\sigma/D}$, $\gamma = 1.0$, K=7

$\bar{\lambda} = \lambda / \sin^2\alpha$

$\bar{\lambda}$ \ α	90°	75°	60°	45°	30°
1	35.94	38.13	46.07	65.81	122.81
2	73.25	72.79	81.71	106.96	180.01
3	73.25	82.50	105.16	149.24	237.48
4	107.61	109.32	119.46	158.03	301.55
5	131.40	138.79	165.13	199.67	310.71
6	132.09	144.91	165.54	231.57	372.58
7	164.08	157.20	186.18	253.87	421.76
8	164.08	183.17	219.63	290.04	476.87
9	210.87	211.51	219.90	299.49	534.13
10	210.87	224.52	267.44	314.50	577.19

TABLE 3. Frequency Parameter $\varphi a^2\sqrt{\sigma/D}$, $\gamma = 1.5$, K=7

$\bar{\lambda} = \lambda / \sin^2\alpha$

$\bar{\lambda}$ \ α	90°	75°	60°	45°	30°
1	60.70	64.57	78.64	114.13	218.84
2	93.57	98.15	114.60	155.22	271.59
3	148.63	153.23	170.52	217.27	351.09
4	149.33	159.05	196.47	280.72	445.00
5	178.71	192.25	236.89	303.02	569.38
6	226.72	222.21	238.77	337.46	582.56
7	230.57	255.20	310.01	377.44	662.59
8	281.82	296.62	313.17	430.85	771.72
9	303.78	304.20	373.66	474.47	792.85
10	310.80	332.25	390.89	544.05	958.00

TABLE 4. Frequency Parameter $\varphi a^2\sqrt{\sigma/D}$, $\gamma = 2.0$, K=7

$\bar{\lambda} = \lambda / \sin^2\alpha$

$\bar{\lambda}$ \ α	90°	75°	60°	45°	30°
1	98.23	104.80	128.71	189.48	370.62
2	126.93	134.01	159.47	223.23	411.24
3	178.53	185.97	213.47	282.98	481.11
4	252.69	259.54	289.27	364.05	586.71
5	255.78	274.71	340.51	459.39	766.29
6	283.30	301.80	361.50	507.23	1005.66
7	331.24	352.07	392.82	553.67	1063.56
8	348.88	354.80	422.51	588.74	1089.07
9	399.65	426.74	503.75	650.22	1165.71
10	474.24	468.24	514.54	727.31	1337.99

The fundamental frequency parameter ($\sigma a^2 \sqrt{\sigma/D}\ \sin^2 a$) for a clamped parallelogram plate having $a = 60^\circ$, $\gamma = 1.0$ is given by Hamada (1959) as 34.788, by Hasgawa (1957) as 34.624, and by Conway and Farnham (1965) as 34.660. The value obtained by this analysis for the same plate is 34.550 for K = 7 as given in Table 1. The ratio of frequencies of a clamped plate to those of a square plate are given in Fig. 2.

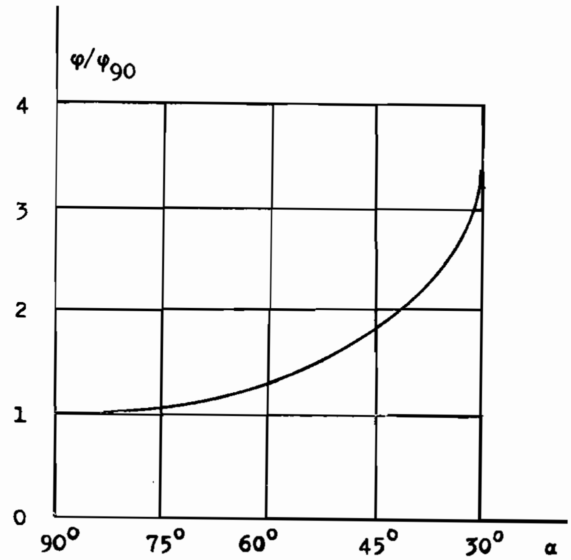

FIG. 2 - Ratio of frequencies of a skew plate to those of a square plate.

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التردد الطبيعي للوحات متوازية الاضلاع المثبتة الحافات

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قسم الرياضيات بجامعة الكويت

خلاصـــــة

يعالج هذا البحث ايجاد طريقة تقريبية للترددات الطبيعية للوحات المرنة التي على شكل متوازي اضلاع والمثبتة تماما عند حافتها . لقد طبقت المعادلة التفاضلية للحركة التي تحققها الازاحة العمودية عند اية نقطة من اللوحة وعبر عن هذه الازاحة بدلالة متسلسلة لا نهائية تحقق تماما الشروط الحدية عند حافة اللوحة وهي انعدام الميل والازاحة ثم طورت طريقة جالركن في جعل الخطأ الناشيء من تعويض الصيغة المقترحة للازاحة في معادلة الحركة نهاية صفرى .