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Four mappings with a common fixed point

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ABSTRACT

It is proved that if S and I are commuting mappings and T and J are commuting mappings of a complete metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) \leq c \cdot \max\{d(Ix,Jy), d(Ix,Ty), d(Sx,Jy)\}$$

for all x, y in X, where $0 \leq c < 1$, if the range of I contains the range of S and the range of J contains the range of T and if I and J are continuous, then S, T, I and J have a unique common fixed point. Other related results are proved.

We prove the following theorem.

Theorem 1. Let S and I be commuting mappings and T and J be commuting mappings of a complete metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) \leq c \cdot \max\{d(Ix,Jy), d(Ix,Ty), d(Sx,Jy)\} \tag{1}$$

for all x, y in X, where $0 \leq c < 1$. If the range of I contains the range of S and the range of J contains the range of T and if I and J are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. Let $x = x_0$ be an arbitrary point in X and define a sequence $\{x_n\}$ inductively by choosing a point x_n such that

$$Sx_{n-1} = Ix_n$$

for $n = 1, 2, \ldots$. This can be done since the range of I contains the range of S.

Let us now suppose that the sequence $\{Sx_n\}$ is unbounded. Then the sequence of real numbers $\{d(Sx_n,Tx)\}$ is unbounded and so there exists an integer n such that

$$(1-c)d(Sx_n,Tx) > cd(Tx,Jx)$$

and

$$d(Sx_n,Tx) > \max\{d(Sx_r,Tx): 0 \leq r < n\}. \tag{2}$$

These inequalities imply that for $0 \leq r \leq n$

$$\begin{aligned} cd(Sx_r,Jx) &\leq c[d(Sx_r,Tx)+d(Tx,Jx)] \\ &< d(Sx_n,Tx) \end{aligned}$$

and so

$$d(Sx_n,Tx) > c\cdot\max\{d(Sx_r,Jx)\colon 0 \leq r \leq n\}. \tag{3}$$

It follows from inequalities (2) and (3) that

$$d(Sx_n,Tx) > c\cdot\max\{d(Sx_{n-1},Jx),\ d(Sx_{n-1},Tx),\ d(Sx_n,\ Jx)\}. \tag{4}$$

However, on using inequality (1) we have

$$\begin{aligned} d(Sx_n,Tx) &\leq c\cdot\max\{d(Ix_n,Jx),\ d(Ix_n,Tx),\ d(Sx_n,Jx)\} \\ &= c\cdot\max\{d(Sx_{n-1},Jx),\ d(Sx_{n-1},Tx),\ d(Sx_n,Jx)\}, \end{aligned}$$

contradicting inequality (4). This contradiction implies that the sequence $\{Sx_n\}$ is bounded.

Similarly let y_0 be an arbitrary point in X and define a sequence $\{y_n\}$ inductively by choosing a point y_n such that

$$Ty_{n-1} = Jy_n$$

for $n=1, 2, \ldots$. This can be done since the range of J contains the range of T. The sequence $\{Ty_n\}$ will of course also be bounded.

Thus

$$M = \sup\{d(Sx_r,Ty_s)\colon r,s = 0, 1, 2, \ldots\}$$

is finite. For arbitrary $\varepsilon>0$ now choose a positive integer N such that

$$c^N M < \varepsilon.$$

It follows that for $m, n \geq N$

$$\begin{aligned} d(Sx_m,Ty_n) &\leq c\cdot\max\{d(Ix_m,Jy_n),\ d(Ix_m,Ty_n),\ d(Sx_m,Jy_n)\} \\ &= c\cdot\max\{d(Sx_{m-1},Ty_{n-1}),\ d(Sx_{m-1},Ty_n),\ d(Sx_m,Ty_{n-1})\} \\ &\leq c\cdot\max\{d(Sx_r,Ty_s)\colon m-1 \leq r \leq m;\ n-1 \leq s \leq n\} \\ &\leq c^2\cdot\max\{d(Sx_r,Ty_s)\colon m-2 \leq r \leq m;\ n-2 \leq s \leq n\} \\ &\leq c^N\cdot\max\{d(Sx_r,Ty_s)\colon m-N \leq r \leq m;\ n-N \leq s \leq n\} \\ &\leq c^N M \\ &< \varepsilon \end{aligned}$$

and so

$$\begin{aligned} d(Sx_m,Sx_r) &\leq d(Sx_m,Ty_n)+d(Ty_n,Sx_r) \\ &< 2\varepsilon \end{aligned}$$

for $m, n, r \geq N$. Thus $\{Sx_n\}=\{Ix_{n+1}\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Further, since

$$d(Sx_n,Ty_n) < \varepsilon$$

for $n \geq N$, the sequence $\{Ty_n\}=\{Jy_{n+1}\}$ also converges to z.

Using the commutativity of S and I and the continuity of I we see that the sequence $\{SIx_n\}=\{I^2x_{n+1}\}$ converges to Iz. By inequality (1) we have

$$d(SIx_n,Ty_n) \leq c\cdot\max\{d(I^2x_n,Jy_n),\ d(I^2x_nTy_n),\ d(SIx_n,Jy_n)\}$$

and so on letting n tend to infinity we see that

$$d(Iz,z) \leq cd(Iz,z).$$

Since $c<1$ it follows that z is a fixed point of I. Further

$$d(Sx,Ty_n) \leq c\cdot\max\{d(Iz,Jy_n),\ d(Iz,Ty_n),\ d(Sz,Jy_n)\}$$

and on letting n tend to infinity we have

$$d(Sz,z) \leq cd(Sz,z).$$

Thus z must also be a fixed point of S.

We can similarly prove that z is a fixed point of T and J and so z is a common fixed point of S, T, I and J.

Now suppose that T and J have a second common fixed point w. Then

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &\leq c\cdot\max\{d(Iz,Jw),\ d(Iz,Tw),\ d(Sz,Jw)\} \\ &= cd(z,w). \end{aligned}$$

Since $c<1$, the common fixed point z of T and J must therefore be unique. Similarly, z is the unique common fixed point of S and I. Thus z must also be the unique common fixed point of S, T, I and J. This completes the proof of the theorem.

Corollary 1. Let S and I be commuting mappings and T and J be commuting mappings of a complete metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) \leq c\cdot\max\{d(Ix,Jy),\ d(Ix,Ty),\ d(Sx,Jy),\ \tfrac{1}{2}d(Ix,Sx),\ \tfrac{1}{2}d(Jy,Ty)\} \quad (5)$$

for all x, y in X, where $0 \leq c < 1$. If the range of I contains the range of S and the range of J contains the range of T and if I and J are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. For all x, y in X we have

$$\begin{aligned} \tfrac{1}{2}d(Ix,Sx) &\leq \tfrac{1}{2}[d(Ix,Ty)+d(Ty,Sx)] \\ &\leq \max\{d(Ix,Jy),\ d(Ix,Ty), d(Sx,Jy),\ d(Ty,Sx)\} \end{aligned}$$

and similarly

$$\tfrac{1}{2}d(Jy,Ty) \leq \max\{d(Ix,Jy),\ d(Ix,Ty),\ d(Sx,Jy),\ d(Ty,Sx)\}.$$

Thus

$$\begin{aligned} d(Sx,Ty) &\leq c\cdot\max\{d(Ix,Jy),\ d(Ix,Ty),\ d(Sx,Jy),\ d(Sx,Ty)\} \\ &= c\cdot\max\{d(Ix,Jy),\ d(Ix,Ty),\ d(Sx,Jy)\} \end{aligned}$$

since $c<1$. The conditions of Theorem 1 are therefore satisfied and the result of the corollary now follows.

By letting $I=J$ be the identity mapping in Corollary 1 we have

Corollary 2. Let S and T be mappings of a complete metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) \leq c\cdot\max\{d(x,y),\, d(x,Ty),\, d(Sx,y),\, \tfrac{1}{2}d(x,Sx),\, \tfrac{1}{2}d(y,Ty)\} \qquad (6)$$

for all x, y in X, where $0 \leq c < 1$. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and T.

The result of this corollary with inequality (6) replaced by the inequality

$$d(Sx,Ty) \leq c\cdot\max\{d(x,y),\, d(x,Ty),\, d(Sx,y)\}$$

was given in Fisher (1980).

Note also that if $S=T$ in Corollary 2 then inequality (6) can be replaced by the inequality

$$d(Tx,Ty) \leq c\cdot\max\{d(x,y),\, d(x,Ty),\, d(Tx,y),\, d(x,Tx),\, d(y,Ty)\},$$

(see Fisher 1979a). Corollary 2 does not, however, hold if the factors $\frac{1}{2}$ occurring in inequality (6) are omitted (see Fisher 1979b).

By letting $S=T$ be the identity mapping in Corollary 1 we have

Corollary 3. Let I and J be continuous mappings of a complete metric space (X,d) onto itself satisfying the inequality

$$d(x,y) \leq c\cdot\max\{d(Ix,Jy),\, d(Ix,y),\, d(x,Jy),\, \tfrac{1}{2}d(Ix,x),\, \tfrac{1}{2}d(Jy,y)\}$$

for all x, y in X, where $0 \leq c < 1$. Then I and J have a unique common fixed point z. Further, z is the unique fixed point of I and J.

We now show that although the mappings S and I and the mappings T and J in Theorem 1 have a unique common fixed point it is possible for them to have other fixed points. To see this let $X=\{0,2^n : n=0,\pm 1,\pm 2,\ldots\}$ with the usual metric for real numbers and define mappings T and I on X by

$$Tx = 0 : x \in X,$$

$$I0 = 0,\ I2^n = 2^{n+1} : n = 0,\pm 1,\pm 2,\ldots$$

and let $S=J$ be the identity mapping. All the conditions of the thorem are satisfied with $c=\frac{1}{2}$ but S and J have an infinite number of common fixed points.

To see that the condition that the mappings T and J commute is necessary in Theorem 1 let $X=\{1,2\}$ with the discrete metric. Define mappings $S=T=I$ and J on X by

$$S1 = S2 = 1, \quad J1 = 2, \quad J2 = 1.$$

All the conditions of the theorem are satisfied with $c=\frac{1}{2}$ except for the commutativity of T and J but J has no fixed points. Note, however, that the proof of Theorem 1 shows that S and I necessarily have a common fixed point even if T and J do not commute.

Next we show that the condition that the range of J contains the range of T is also necessary in Theorem 1. To see this let X be the set of real numbers $x \geq 1$ with the usual metric for real numbers. Define mappings $S=I$, T and J on X by

$$Sx = 1, \quad Tx = x, \quad Jx = 2x\colon x \in X.$$

All the conditions of the theorem are satisfied with $c = \frac{1}{2}$ except that the range of J does not contain the range of T but J has no fixed points.

The condition that J be continuous is also necessary in Theorem 1. To see this let X be the closed interval [0, 1] with the usual metric for real numbers. Define mappings $S = I$, T and J on X by

$$Sx = 0\colon x \in X,$$
$$T0 = \tfrac{1}{2},\ Tx = \tfrac{1}{4}x\colon x \in X, x \neq 0,$$
$$J0 = 1,\ Jx = \tfrac{1}{2}x\colon x \in X, x \neq 0.$$

All the conditions of the theorem are satisfied with $c = \frac{1}{2}$ except that J is not continuous but T and J have no fixed points. The proof of Theorem 1 again shows that S and I necessarily have a common fixed point even if J is not continuous.

Theorem 2. Let S and I be commuting mappings and T and J be commuting mappings of a complete metric space (X,d) into itself satisfying inequality (5) for all x, y in X, where $0 \leq c < 1$. If the range of I contains the range of S and the range of J contains the range of T and if S and T are continuous, then S and T have a common fixed point z.

Proof. Define the sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 1. Since inequality (5) implies inequality (1) it follows that the sequences $\{Sx_n\} = \{Ix_{n+1}\}$ and $\{Ty_n\} = \{Jy_{n+1}\}$ converge to a point z in X. Since the mappings S and I commute and S is continuous it follows that the sequence $\{S^2x_n\} = \{ISx_{n+1}\}$ converges to the point Sz. We have

$$d(S^2x_n,Ty_n\} \leq c\cdot\max\{d(ISx_n,Jy), d(ISx_n,Ty_n), d(S^2x_n,Jy_n), \tfrac{1}{2}d(ISx_n,S^2x_n), \tfrac{1}{2}d(Jy_n,Ty_n)\}$$

and on letting n tend to infinity we have

$$d(Sz,z) \leq cd(Sz,z).$$

It follows that z is a fixed point of S.

We can prove similarly that z is also a fixed point of T. This completes the proof of the theorem.

It seems possible that the mappings S, T, I and J in Theorem 2 must necessarily have a common fixed point although I suspect that this is not true in general. However, if the metric space is bounded then S, T, I and J must have a common fixed point as is given in the next theorem.

Theorem 3. Let S and I be commuting mappings and T and J be commuting mappings of a complete, bounded metric space (X,d) into itself satisfying inequality (5) for all x, y in X, where $0 \leq c < 1$. If the range of I contains the range of S and the range of J contains the range of T and if S and T are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. Theorem 2 tells us that S and T have a common fixed point z. Then for $r, s = 0, 1, 2, \ldots$

$$\begin{aligned} d(I^r z, J^s z) &= d(SI^r z, TJ^s z) \\ &\leq c\cdot\max\{d(I^{r+1}z, J^{s+1}z),\ d(I^{r+1}z, TJ^s z),\ d(SI^r z,\ J^{s+1}z)\} \\ &= c\cdot\max\{d(I^{r+1}z, J^{s+1}z),\ d(I^{r+1}z, J^s z),\ d(I^r z, J^{s+1}z)\} \\ &\leq c\cdot\max\{d(I^{r'}z, J^{s'}z):\ r', s' = 0, 1, 2, \ldots\} \end{aligned}$$

and so

$$\begin{aligned} d(I^r z, J^s z) &\leq c^n\cdot\max\{d(I^{r'}z, J^{s'}z):\ r', s' = 0, 1, 2, \ldots\} \\ &\leq c^n M \end{aligned}$$

for $n=1, 2, \ldots$, where

$$M = \sup\{d(x,y):\ x, y \in X\}$$

is finite since X is bounded. In particular

$$d(z, Jz) \leq c^n M$$

and on letting n tend to infinity it follows that z is a fixed point of J. Similarly, z is a fixed point of I. The uniqueness of z follows as in the proof of Theorem 1. This completes the proof of the theorem.

The proof of the following corollary follows easily on letting $S=T$ be the identity mapping in Theorem 3.

Corollary. Let I and J be mappings of a complete, bounded metric space (X,d) onto itself satisfying the inequality

$$d(x,y) \leq c\cdot\max\{d(Ix,Jy),\ d(Ix,y),\ d(x,Jy),\ \tfrac{1}{2}d(Ix,x),\ \tfrac{1}{2}d(Jy,y)\}$$

for all x, y in X, where $0 \leq c < 1$. Then I and J have a unique common fixed point z. Further, z is the unique fixed point of I and J.

Theorem 4. Let S and I be commuting mappings and T and J be commuting mappings of a complete metric space (X,d) into itself satisfying inequality (5) for all x, y in X, where $0 \leq c < 1$. If the range of I contains the range of S and the range of J contains the range of T, if S and T are continuous and if the number of fixed points of S and T is finite, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. Theorem 2 tells us that S and T have a common fixed point z_1. Let $Z=\{z_1, \ldots, z_n\}$ be the set of fixed points of S and let $Z'=\{z_1=z_1', z_2', \ldots, z_m'\}$ be the set of fixed points of T. Now

$$Iz_r = ISz_r = SIz_r$$

and so Iz_r is a fixed point of S for $r=1, \ldots, n$. Thus I maps Z into Z. Similarly J maps Z' into Z'. Then for $r=1, \ldots, n$ and $s=1, \ldots, m$

$$\begin{aligned} d(Iz_r, Jz_s') &= d(SIz_r, TJz_s') \\ &\leq c\cdot\max\{d(I^2 z_r, J^2 z_s'),\ d(I^2 z_r, TJz_s'),\ d(SIz_r, J^2 z_s')\} \\ &\leq c\cdot\max\{d(Iz_{r'}, Jz_{s'}'):\ 1 \leq r' \leq n;\ 1 \leq s' \leq m\} \end{aligned}$$

and so

$$d(Iz_r,Jz'_s) \leq c^n \cdot \max\{d(Iz_{r'},Jz'_{s'}) \colon 1 \leq r' \leq n;\ 1 \leq s' \leq m\}$$

for $n = 1, 2, \ldots$. Letting n tend to infinity it follows that

$$Iz_r = Jz'_s$$

for $r = 1, \ldots, n$ and $s = 1, \ldots, m$. Putting $Iz_r = Jz'_s = z$, where z must be in $Z \cap Z'$, we see that $z = z_r$ for some r and $z = z'_s$ for some s. It follows that z is a common fixed point of S, T, I and J. The uniqueness of z follows as before. This completes the proof of the theorem.

We finally prove a theorem for compact metric spaces.

Theorem 5. Let S and I be commuting mappings and T and J be commuting mappings of a compact metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) < \max\{d(Ix,Jy), d(Ix,Ty), d(Sx,Jy), \tfrac{1}{2}d(Ix,Sx), \tfrac{1}{2}d(Jy,Ty)\} \qquad (7)$$

for all x, y in X for which the right-hand side of the inequality is positive. If the range of I contains the range of S and the range of J contains the range of T and if S, T, I and J are continuous, then S, T, I and J have a unique common fixed point z. Further, z is the unique common fixed point of S and I and of T and J.

Proof. Suppose first of all that the right-hand side of inequality (7) is positive for all x, y in X. Then the function

$$f(x,y) = \frac{d(Sx,Ty)}{\max\{d(Ix,Jy), d(Ix,Ty), d(Sx,Jy), \tfrac{1}{2}d(Ix,Sx), \tfrac{1}{2}d(Jy,Ty)\}}$$

is a continuous function defined on the compact metric space $X \times X$ and so achieves its maximum value c. Inequality (7) implies that $c < 1$ and it follows that the conditions of Theorem 4 are satisfied. Hence S, T, I and J have a common fixed point z.

Now suppose that the right-hand side of inequality (7) is zero for some x, y in X. Then

$$Ix = Jy = Sx = Ty$$

and so

$$SIx = S^2x = ISx.$$

If $S^2x \neq Ty$ then

$$\begin{aligned} d(S^2x,Ty) &< \max\{d(ISx,Jy), d(ISx,Ty), d(S^2x,Jy)\} \\ &= d(S^2x,Ty) \end{aligned}$$

giving a contradiction. It follows that

$$S^2x = Ty = STy$$

and so $Ty = z$ is a fixed point of S. Further

$$Iz = ITy = ISx = SIx = STy = Sz = Z$$

and it follows that z is a common fixed point of S and I.

Similarly, we can prove that T and J have a common fixed point w. If $z \neq w$ we have

$$\begin{aligned} d(z,w) &= d(Sz,Tw) \\ &< \max\{d(Iz,Jw),\ d(Iz,Tw),\ d(Sz,Jw)\} \\ &= d(z,w) \end{aligned}$$

giving a contradiction. It follows that $z = w$ is a common fixed point of S, T, I and J and that z is the unique common fixed point of S and I and of T and J. This completes the proof of the theorem.

The following corollaries follow easily.

Corollary 1. Let S and T be continuous mappings of a compact metric space (X,d) into itself satisfying the inequality

$$d(Sx,Ty) < \max\{d(x,y),\ d(x,Ty),\ d(Sx,y),\ \tfrac{1}{2}d(x,Sx),\ \tfrac{1}{2}d(y,Ty)\}$$

for all x, y in X for which the right-hand side of the inequality is positive. Then S and T have a unique common fixed point z. Further, z is the unique fixed point of S and T.

Corollary 2. Let I and J be continuous mappings of a compact metric space (X,d) onto itself satisfying the inequality

$$d(x,y) < \max\{d(Ix,Jy),\ d(Ix,y),\ d(x,Jy),\ \tfrac{1}{2}d(Ix,x),\ \tfrac{1}{2}d(Jy,y)\}$$

for all x, y in X for which the right-hand side of the inequality is positive. Then I and J have a unique common fixed point z. Further, z is the unique fixed point of I and J.

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أربعة تطبيقات ذوات نقطة ثابتة مشتركة

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خلاصة

لقد تم اثبات النتيجة الآتية :
اذا كان I , S تطبيقين ابداليين وكان T , J تطبيقين ابداليين أيضا من فضاء متري تام (X , d) إلى نفسه وكانت المتباينة

$$d(Sx,Ty) \leq c \cdot \max\{d(Ix,Jy),\, d(Ix,Ty),\, d(Sx,Jy)\}$$

لجميع x , y في X حيث صفر $\leq c < 1$ وكان مدى I يحتوي على مدى S ومدى J يحتوي على مدى T وكان كل من J, I متصلا ، فان للتطبيقات S, T, I, J نقطة وحيدة مشتركة وثابتة . كما تم اثبات نتائج ذات علاقة بها .