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Application of absolute matrix summability to Fourier and allied series

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ABSTRACT

In this paper we investigate some of the applications of an absolutely conservative series-to-series transformation to the Fourier series and its conjugate series with factors. Consequently we improve some of the earlier results (Pati 1959; Dikshit 1970) and obtain some new results.

1. DEFINITIONS AND NOTATIONS

Let $\Sigma a'_m$ be an infinite series and $T = (\alpha_{n,m})$ be an $\infty \times \infty$ matrix. The series-to-series transformation

$$b'_n = \sum_{m=0}^{\infty} \alpha_{n,m}\, a'_m, \tag{1.1}$$

where b'_n is supposed to exist for every $n = 0, 1, 2, \ldots$, defines the matrix transformation of the series $\Sigma a'_m$. If $\Sigma b'_n$ converges to s then $\Sigma a'_m$ is said to be T-summable to s. We say that $\Sigma a'_m$ is absolutely summable T or summable $|T|$, if $\Sigma b'_n$ converges absolutely.

We shall consider the case in which T is absolutely conservative, i.e. whenever $\Sigma a'_m$ converges absolutely so does $\Sigma b'_n$. It is known (see Mears 1937; Knopp & Lorentz 1949) that in order that this should hold it is necessary and sufficient that, $m \geqslant 0$

$$\sum_{n=0}^{\infty} |\alpha_{n,m}| = O(1). \tag{1.2}$$

In order that T should be absolutely regular, i.e. whenever $\Sigma a'_m$ converges absolutely then $\Sigma b'_n$ converges absolutely to the same sum, it is necessary and sufficient that (1.2) should hold and that further, for all $m \geqslant 0$,

$$\sum_{n=0}^{\infty} \alpha_{n,m} = 1. \tag{1.3}$$

Let $f(t)$ be a periodic function with period 2π and integrable-L over $(-\pi,\pi)$. We assume without any loss of generality that the constant term in the Fourier series of $f(t)$ is zero. Thus

$$\int_{-\pi}^{\pi} f(t)\, dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=1}^{\infty} A_n(t).$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\}$$
$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\}.$$

We will suppose throughout that $0 < t \leqslant \pi$ and apply the hypotheses (2.3) and (2.4) with $K = [\pi/t]$. To avoid complicated suffixes we write $\alpha(n,m)$ whenever n,m are replaced by more complicated expressions. For any sequence $\{g(n,m)\}$, we also write

$$\Delta_m g(n,m) = g(n,m) - g(n,m+1).$$

2. INTRODUCTION

We will be concerned with consequences of the assumption

$$\int_0^{\pi} t^{-\beta} |d\,\phi(t)| < \infty (0 \leqslant \beta < 1) \tag{2.1}$$

or

$$\text{(i) } \psi(+0) = 0, \qquad \text{(ii) } \int_0^{\pi} t^{-\beta} |d\,\psi(t)| < \infty (0 < \beta < 1). \tag{2.2}$$

We state the following result concerning the absolute Cesàro summability:

Theorem A. Let $\delta > \beta$. Then

$$(2.1) \Rightarrow \Sigma\, n^{\beta} A_n(x) \in |C,\delta|$$

and

$$(2.2) \Rightarrow \Sigma\, n^{\beta} B_n(x) \in |C,\delta|.$$

The case $\beta = 0$ for the Fourier series is due to Bosanquet (1936); and the case $0 < \beta < 1$ of Theorem A is due to Mohanty (1950).

In this paper, we generalise Theorem A for absolute matrix summability of $\Sigma n^{\beta} A_n(x)$ and $\Sigma n^{\beta} B_n(x)$. The case $\beta = 0$ of our theorem for the Fourier series corresponds to a result due to Kuttner & Sahney (1972).

We prove the following:

Theorem. Let $0 \leqslant \beta < 1$. Let $T = (\alpha_{n,m})$ be an absolutely conservative series-to-series transformation with $\alpha_{n,m} \geqslant 0$ for all n,m. Suppose, for each fixed n, there exists a positive integer either
(a) $r(n)$ such that $\{m^{\beta}\alpha_{n,m}\} \nearrow$ for $1 \leqslant m \leqslant r(n)$ and $\searrow$ for $m \geqslant r(n)$ and that

$$\sum_{r(n) \geqslant 2K} [r(n)]^{-1} \sum_{m=r(n)-K}^{r(n)+K} m^{\beta}\alpha_{n,m} = O(K^{\beta})\,(K \geqslant 1) \tag{2.3}$$

or

(b) $s(n)$ such that $\{m^{\beta-1}\alpha_{n,m}\}\downarrow$ for $1 \leqslant m \leqslant s(n)$ and $\downarrow$ for $m \geqslant s(n)$ and that

$$\sum_{s(n) \geqslant 2K} [s(n)]^{-1} \sum_{m=s(n)-K}^{s(n)+K} m^{\beta}\alpha_{n,m} = O(K^{\beta})\ (K \geqslant 1). \tag{2.4}$$

Then

$$(2.1) \Rightarrow \Sigma\, n^{\beta} A_n(x) \in |T|. \tag{2.5}$$

And, whenever (a) or (b) holds for $0 < \beta < 1$,

$$(2.2) \Rightarrow \Sigma\, n^{\beta} B_n(x) \in |T|. \tag{2.6}$$

Remark. It is clear that (2.3) and (2.4), respectively, are equivalent to

$$\sum_{r(n) \geqslant 2K} \sum_{m=r(n)-K}^{r(n)+K} m^{\beta-1}\alpha_{n,m} = O(K^{\beta}) \tag{2.3'}$$

and

$$\sum_{s(n) \geqslant 2K} \sum_{m=s(n)-K}^{s(n)+K} m^{\beta-1}\alpha_{n,m} = O(K^{\beta}). \tag{2.4'}$$

3. LEMMAS

For the proof of the theorem we require the following lemmas:

Lemma 1. Let $T = (\alpha_{n,m})$ be an absolutely conservative series-to-series transformation and let, for every fixed n and $0 \leqslant \beta < 1$, $\{m^{\beta-1}\alpha_{n,m}\}$ be ultimately non-negative and non-increasing. Then

$$t^{\beta} \sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m} \sin mt \qquad (0 \leqslant \beta < 1) \tag{3.1}$$

and

$$t^{\beta} \sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m} \cos mt \qquad (0 < \beta < 1) \tag{3.2}$$

are boundedly convergent in $0 < t < \pi$.

Proof. The convergence of (3.1) for every fixed t in $0 < t < \pi$ follows from Dirichlet's test.

Now, let M be a positive constant such that $\{m^{\beta-1}\alpha_{n,m}\}$ is non-negative and non-increasing for $m \geqslant M$. Then we have, uniformly in m_1, m_2 for $K, M \leqslant m_1 \leqslant m_2$

$$\left| t^{\beta} \sum_{m=m_1}^{m_2} m^{\beta-1}\alpha_{n,m} \sin mt \right| \leqslant \frac{t^{\beta} m_1^{\beta-1}\alpha(n,m_1)}{2 \sin \frac{1}{2}t}.$$

But (1.2) implies that $\alpha(n,m)$ is bounded; hence the expression on the right of the above inequality is O(1) uniformly in the range considered.

Since M is a constant,

$$t^{\beta} \sum_{m=1}^{M-1} |m^{\beta-1}\alpha_{n,m} \sin mt|$$

is bounded; also, if $K \geqslant M$,

$$\begin{aligned} \Sigma &= t^{\beta} \sum_{m=M}^{K} m^{\beta-1}\alpha_{n,m} \sin mt| \\ &\leqslant t^{1+\beta} \sum_{m=M}^{K} m^{\beta}\alpha_{n,m} \\ &= O(1), \end{aligned}$$

by the boundedness of $\alpha_{n,m}$ and the definition K.

This proves that

$$\sum_{m=1}^{M} m^{\beta-1}\alpha_{n,m}\, t^{\beta} \sin mt$$

is bounded uniformly in M and t. Consequently, it proves (3.1).

For the proof of (3.2), we proceed as in (3.1) and in dealing with Σ we use

$$|\cos mt| \leqslant 1.$$

Note that, for this argument, it is essential that $\beta > 0$.

This completes the proof of the lemma.

Lemma 2 (Kuttner & Sahney 1972). Suppose that θ_m is non-negative and non-decreasing for $1 \leqslant m \leqslant s$ and non-increasing for $m \geqslant s$. Then for any positive integer a, b and any t with $0 < t \leqslant \pi$

$$|\sum_{m=a}^{b} \theta_m e^{imt}| \leqslant M \sum_{m=\max(1,\, s-k)}^{s+k} \theta_m, \tag{3.3}$$

where M is an absolute constant.

4. PROOF OF THE THEOREM

Proof of (2.5). We have

$$\begin{aligned} n^{\beta}A_n(x) &= \frac{2}{\pi}\int_0^{\pi} \phi(t) n^{\beta} \cos nt\, dt \\ &= -\frac{2}{\pi}\int_0^{\pi} n^{\beta-1} \sin nt\, d\,\phi(t), \end{aligned}$$

integrating by parts. We write

$$u_n = \sum_{m=1}^{\infty} m^{\beta}\alpha_{n,m}A_m(x),$$

so that we have to prove that

$$\sum_{n=0}^{\infty} |u_n| < \infty.$$

Then

$$u_n = -\frac{2}{\pi}\sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m}\int_0^{\pi}\sin mt\, d\,\phi(t)$$
$$= -\frac{2}{\pi}\int_0^{\pi}\left\{\sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m}\sin mt\right\}d\,\phi(t),$$

by Lemma 1. Hence, in view of (2.1), it is enough to prove that

$$\Sigma = \sum_{n=0}^{\infty}\left|\sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m}\sin mt\right|$$
$$= O(t^{-\beta}).$$

First we consider those values for n for which $r(n) < 2K$ in case (a) and $s(n) < 2K$ in case (b). Since in case (b) $m^{\beta-1}\alpha_{n,m}$ is non-increasing for $m \geqslant 2K$ and in case (a) $m^{\beta}\alpha_{n,m}$ is non-increasing for $m \geqslant 2K$, it follows that $m^{\beta-1}\alpha_{n,m}$ is non-increasing for $m \geqslant 2K$ in both cases (a) and (b). Therefore, if Σ^* denotes a sum taken over those values of n (if any) for which $r(n) < 2K$ in case (a) or $s(n) < 2K$ in case (b), we write

$$\Sigma = \sum{}^{*}\left|\left(\sum_{m=1}^{2K-1} + \sum_{m=2K}^{\infty}\right)\alpha_{n,m} m^{\beta-1}\sin mt\right|$$
$$\leqslant \Sigma_1 + \Sigma_2, \text{ say.}$$

However

$$\Sigma_1 \leqslant \sum{}^{*}\sum_{m=1}^{2K-1} m^{\beta-1}\alpha_{n,m}\,|\sin mt|$$
$$\leqslant t\sum{}^{*}\sum_{m=1}^{2K-1} m^{\beta}\alpha_{n,m}$$
$$= O(t^{1-\beta})\sum_{m=1}^{2K-1}\sum_{n=0}^{\infty}\alpha_{n,m}$$
$$= O(t^{-\beta}),$$

by (1.2).

$$\Sigma_2 \leqslant \sum{}^{*}\left|\sum_{m=2K}^{\infty} m^{\beta-1}\alpha_{n,m}\sin mt\right|$$
$$\leqslant \sum{}^{*}(2K)^{\beta-1}\alpha_{n,2K}\max_{2K\leqslant u\leqslant\infty}\left|\sum_{m=2K}^{u}\sin mt\right|$$
$$\leqslant \sum_{n=0}^{\infty}(2K)^{\beta-1}\alpha_{n,2K}\cdot O(t^{-1})$$
$$= O(t^{-\beta})\sum_{n=0}^{\infty}\alpha_{n,2K}$$
$$= O(t^{-\beta}),$$

by (1.2).

Now we consider those values of n for which $r(n) \geqslant 2K$ or $s(n) \geqslant 2K$.

First we consider case (b). For any fixed n, we apply Lemma 2 with $\theta_m = m^{\beta-1}\alpha_{n,m}$ and take imaginary part of (3.3). Then inner sum of Σ does not exceed

$$M \sum_{m=s(n)-K}^{s(n)+K} m^{\beta-1}\alpha_{n,m}.$$

Hence

$$\begin{aligned}\sum_{s(n)\geqslant 2K} &= \mathrm{O}\Big\{ \sum_{s(n)\geqslant 2K} \sum_{m=s(n)-K}^{s(n)+K} m^{\beta-1}\alpha_{n,m}\Big\}\\ &= \mathrm{O}(t^{-\beta}),\end{aligned}$$

by (2.4′).

Now consider case (a). Again since $m^{\beta}\alpha_{n,m}$ is non-increasing for $m \geqslant r(n)$ so is $m^{\beta-1}\alpha_{n,m}$.

$$\begin{aligned}\Sigma &= \sum_{r(n)\geqslant 2K} \Big| \sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m} \sin mt\Big|\\ &= \sum_{r(n)\geqslant 2K} \Big|\Big(\sum_{m=1}^{K-1} + \sum_{m=K}^{r(n)-K} + \sum_{m=r(n)-K+1}^{\infty}\Big) m^{\beta-1}\alpha_{n,m} \sin mt\Big|.\end{aligned}$$

The proof of the first part of the sum for which $m < K$ can be treated in the same way as Σ_1.

However

$$\begin{aligned}&\Big| \sum_{m=r(n)-K+1}^{\infty} m^{\beta-1}\alpha_{n,m} \sin mt\Big|\\ &\leqslant \sum_{m=r(n)-K}^{r(n)+K} m^{\beta-1}\alpha_{n,m} + [r(n)+K)^{\beta-1}\alpha_{n,r(n)+K}\cdot \mathrm{O}(t^{-1})\\ &= \sum_{m=r(n)-K}^{r(n)+K} m^{\beta-1}\alpha_{n,m} + \mathrm{O}(1/Kt) \sum_{m=r(n)}^{r(n)+K} m^{\beta-1}\alpha_{n,m}\\ &= \mathrm{O}(1) \sum_{m=r(n)-K}^{r(n)+K} m^{\beta-1}\alpha_{n,m},\end{aligned}$$

therefore, proceeding as in case (b), the last part of the sum does not exceed $\mathrm{O}(t^{-\beta})$. And the second part of the sum is

$$\sum_{r(n)\geqslant 2K} R_n(t),$$

where

$$\begin{aligned}R_n(t) &= \frac{1}{2\sin\frac{1}{2}t}\Big| \sum_{m=K}^{r(n)-K} \frac{\alpha_{n,m}}{m^{1-\beta}} \{\cos(m-\tfrac{1}{2})t\text{-}\cos(m+\tfrac{1}{2})t\}\Big|\\ &= \frac{1}{2\sin\frac{1}{2}t}\Big| - \sum_{m=K}^{r(n)-K} \cos(m+\tfrac{1}{2})t\, \Delta_m(m^{\beta-1}\alpha_{n,m}) +\end{aligned}$$

$$+K^{\beta-1}\alpha_{n,K}\cos(K-\tfrac{1}{2})t-$$

$$-(r(n)-K+1)^{\beta-1}\alpha[n,r(n)-K+1]\cos[r(n)-K+\tfrac{1}{2}]t\Bigg|,$$

by Abel's transformation. And, since

$$\Delta_m\left(\frac{\alpha_{n,m}}{m^{1-\beta}}\right)=\frac{m^\beta\alpha_{n,m}}{m(m+1)}+\frac{\Delta_m(m^\beta\alpha_{n,m})}{m+1}, \tag{4.1}$$

we have

$$R_n(t)=\mathrm{O}\left[t^{-1}\left\{\sum_{m=K}^{r(n)-K}\frac{m^\beta\alpha_{n,m}}{m(m+1)}+\sum_{m=K}^{r(n)-K}\frac{|\Delta_m(m^\beta\alpha_{n,m})|}{m+1}+\right.\right.$$

$$\left.\left.+K^{\beta-1}\alpha_{n,K}+\frac{\alpha[n,r(n)-K+1]}{[r(n)-K+1]^{1-\beta}}\right\}\right]$$

$$=\mathrm{O}[R_n^1(t)+R_n^2(t)+R_n^3(t)+R_n^4(t)],\text{ say.}$$

Since $\{m^\beta\alpha_{n,m}\}\downarrow$ for $m\leqslant r(n)$

$$R_n^2(t)=-t^{-1}\sum_{m=K}^{r(n)-K}\frac{\Delta_m(m^\beta\alpha_{n,m})}{m+1}$$

$$=t^{-1}\left\{\sum_{m=K}^{r(n)-K}\frac{m^\beta\alpha_{n,m}}{m(m+1)}-\frac{K^\beta\alpha_{n,K}}{K}+\right.$$

$$\left.+\frac{\alpha[n,r(n)-K+1]}{[r(n)-K+1]^{1-\beta}}\right\}\text{ (by Abel's transformation)}$$

$$=R_n^1(t)-R_n^3(t)+R_n^4(t).$$

So that $R_n(t)=\mathrm{O}\{R_n^1(t)+R_n^4(t)\}$

and therefore

$$\sum_{r(n)\geqslant 2K}R_n(t)=\mathrm{O}\left\{\sum_{r(n)\geqslant 2K}[R_n^1(t)+R_n^4(t)]\right\}.$$

Now

$$\sum_{r(n)\geqslant 2K}R_n^1(t)<t^{-1}\sum_{m=K}^{\infty}m^{\beta-2}\sum_{n=0}^{\infty}\alpha_{n,m}$$

$$\leftarrow=\mathrm{O}(t^{-\beta}),$$

by (1.2), uniformly in $0<t<\pi$. And

$$R_n^4(t)=\mathrm{O}\left\{t^{-1}\frac{\alpha[n,r(n)-K+1]}{r(n)}[r(n)-K+1]^\beta\right\}$$

$$= O(1)\,[Kr(n)t]^{-1} \sum_{m=r(n)-K+1}^{r(n)} m^{\beta}\alpha_{n,m}$$

$$= O(1)\left\{[r(n)]^{-1} \sum_{m=r(n)-K+1}^{r(n)} m^{\beta}\alpha_{n,m}\right\}$$

so that, by (2.3),

$$\sum_{r(n)\geqslant 2K} R_n^4(t) = O(t^{-\beta}),$$

uniformly in $0 < t < \pi$.

This completes the proof of (2.5).

Proof of (2.6). We have

$$n^{\beta}B_n(x) = \frac{2}{\pi}\int_0^{\pi} \psi(t)n^{\beta}\sin nt\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} n^{\beta-1}\cos nt\,d\psi(t),$$

integrating by parts and using $\psi(+0) = 0$. Thus, proceeding as in (2.5) and using Lemma 1 and (2.2), it is sufficient to show that

$$\Sigma = \sum_{n=0}^{\infty}\left|\sum_{m=1}^{\infty} m^{\beta-1}\alpha_{n,m}\cos mt\right| \tag{4.2}$$

$$= O(t^{-\beta}).$$

Proceeding as in (2.5), first we consider those values of n for which $r(n) < 2K$ in case (a) and $s(n) < 2K$ in case (b).

$$\sum_{r(n)\text{ or }s(n)<2/K} = \sum_{n=0}^{\infty}\left|\left(\sum_{m=1}^{2k-1} + \sum_{m=2K}^{\infty}\right) m^{\beta-1}\alpha_{n,m}\cos mt\right|$$

$$\leqslant \varepsilon_1 + \varepsilon_2, \text{ say.}$$

However

$$\varepsilon_1 \leqslant \sum_{n=0}^{\infty}\sum_{m=1}^{2K-1} m^{\beta-1}\alpha_{n,m}$$

$$= \sum_{m=1}^{2K-1} m^{\beta-1}\sum_{n=0}^{\infty}\alpha_{n,m}$$

$$= O(t^{-\beta}),$$

by (1.2). And

$$\varepsilon_2 = O(1),$$

follows on proceeding as in Σ_2 of (2.5).

That part of the sum in (4.2) which is taken over those values of n for which $r(n) \geqslant 2K$ or $s(n) \geqslant 2K$ may be dealt with in the same way as the corresponding part of the sum in (2.5), with $\sin mt$ replaced by $\cos mt$.

5. REMARKS

We now consider an application of our general theorem to the special case of Nörlund summability. Given the sequence (p_n),

$$t_n = \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m, \tag{5.1}$$

where

$$P_n = \sum_{r=0}^{n} P_r \neq 0,\ P_{-1} = p_{-1} = 0,$$

defines the Nörlund mean (N,p) of the sequence (s_n) generated by the sequence of constants (p_n).

If we write

$$t_n = b'_0 + b'_1 + \ldots + b'_n;$$

$$s_m = a'_0 + a'_1 + \ldots + a'_m$$

we see that (5.1) can be expressed as the series-to-series transformation

$$b'_0 = a'_0;$$

$$b'_n = t_n - t_{n-1} = \sum_{m=1}^{n} \left\{ \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right\} a'_m \ (n \geqslant 1).$$

Thus we have, with the notation of our theorem $\alpha_{n,m} = 0$ for $m > n$, while for $1 \leqslant m \leqslant n$

$$\alpha_{n,m} = \frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}}$$

$$= \frac{P_n p_{n-m} - P_{n-m} p_n}{P_n P_{n-1}}. \tag{5.2}$$

We consider the case in which (p_n) is non-negative and $\uparrow$. Since $P_0 \neq 0$ we then have $p_0 > 0$. Further (since $p_n \geqslant 0$) (P_n) is $\downarrow$, thus it follows from (5.2) that $\alpha_{n,m} \geqslant 0$. Thus after omitting the modulus sign in (1.2) it is easy to see that (1.3) and hence (1.2) holds. Thus in the case now considered (N,p) is absolutely regular. Further, for fixed n, p_{n-m} is non-decreasing and P_{n-m} is non-increasing as m increases from 1 to n. Since $\alpha_{n,m} = 0$ for $m > n$, it follows that condition (a) is satisfied with $r(n) = n$ and eqn. (2.3) becomes

$$\sum_{n=2K}^{\infty} \frac{1}{n\, P_n\, P_{n-1}} \sum_{m=n-K}^{n} m^{\beta}(P_n p_{n-m} - P_{n-m} p_n) = O(K^{\beta}). \tag{5.3}$$

The inner sum in (5.3) does not exceed

$$n^{\beta} \sum_{m=n-k}^{n} P_n p_{n-m} = n^{\beta} P_n P_K.$$

Thus a sufficient condition for (5.3) to hold is that

$$\sum_{n=2K}^{\infty} n^{\beta-1}/P_{n-1} = O(K^{\beta}/P_k) \tag{5.4}$$

which is equivalent to

$$\sum_{n=K}^{\infty} n^{\beta-1}/P_n = O(K^{\beta}/P_K). \tag{5.5}$$

Thus our theorem includes the following result:

Theorem B. Let $0 \leqslant (p_n) \uparrow$ and let (5.5) hold for $0 \leqslant \beta < 1$. Then

$$(2.1) \Rightarrow \Sigma\, n^{\beta} A_n(x) \in |N, p| \; (0 \leqslant \beta < 1) \tag{5.6}$$

and

$$(2.2) \Rightarrow \Sigma\, n^{\beta} B_n(x) \in |N, p| \; (0 < \beta < 1). \tag{5.7}$$

Theorem B includes the case $0 < \beta < 1$ of Theorem A.

The case $\beta = 0$ of (5.6) improves the following due to Dikshit (1970, Theorem 2):

Theorem C. Let $p_n > 0$ and that $(p_{n+1}/p_n) \downarrow$ and less than or equal to 1 for all n. Suppose (5.5) for $\beta = 0$ holds. Then $\Sigma\, A_n(x) \in |N, p|$ whenever (2.1) holds for $\beta = 0$.

The case $\beta = 0$ of (5.6) also improves a result due to Pati (1959).

In view of the following lemma the hypothesis (5.5) may be replaced by

$$\sum_{n=0}^{m} \frac{P_n}{(n+1)^{1+\beta}} = O\left(\frac{P_m}{(m+1)^{\beta}}\right) (0 \leqslant \beta < 1), \tag{5.8}$$

whenever $p_0 > 0$, and hence Theorem B includes a result due to Singh (1964):

Lemma 3. Suppose $p_0 > 0$ and $p_n \geqslant 0$. Let $P(\beta)$ and $Q(\beta)$ respectively denote the assertions that (5.5) and (5.8) hold for $\beta \geqslant 0$ and let $R(\beta)$ denote the assertion for $\beta \geqslant 0$:

There is a constant integer $r > 1$ and a constant $\lambda > 1$ such that, for all sufficiently large n

$$P_{rn} \geqslant r^{\beta} \lambda\, P_n. \tag{5.9}$$

Then $P(\beta)$ and $Q(\beta)$ are equivalent to one another. In fact, either is equivalent to $R(\beta)$.

Proof. The proof of the lemma for $\beta = 0$ is due to Kuttner & Sahney (1972). Thus, we prove the lemma for $\beta > 0$.

Let $P(\beta)$ hold. Then we first show that there is a constant $c > 0$ such that, for all $k \geqslant n \geqslant 1$,

$$\frac{P_k}{k^{\beta}} \geqslant c \frac{P_n}{n^{\beta}}. \tag{5.10}$$

Since P_n is non-decreasing, it follows that (5.10) holds (with $c = 2^{-\beta}$) when $k \leqslant 2n$, so we may suppose that $k > 2n$. Let M be a positive constant such that, for all sufficiently large n

$$\sum_{\nu=n}^{\infty} \frac{1}{\nu^{1-\beta} P_\nu} \leqslant M \frac{n^\beta}{P_n}.$$

Then

$$\frac{n^\beta}{P_n} \geqslant \frac{1}{M} \sum_{\nu=n}^{\infty} \frac{1}{\nu^{1-\beta} P_\nu} \geqslant \frac{1}{M} \sum_{\nu=n}^{k} \frac{1}{\nu^{1-\beta} P_\nu}$$

$$\geqslant \frac{1}{M} \left\{ \frac{1}{k^\beta} \sum_{\nu=n}^{k} \frac{1}{\nu^{1-\beta}} \right\} \frac{k^\beta}{P_k}.$$

But, for $n \geqslant 1$, $k > 2n$, the expression in curly brackets is greater than some positive constant; and (5.10) follows.

Arguing as in Kuttner & Sahney (1972) (lemma 4), we have

$$\frac{n^\beta}{P_n} \geqslant \frac{1}{M} \sum_{k=n}^{rn} \frac{1}{k^{1-\beta} P_k} > \frac{c(rn)^\beta}{M\, P_{rn}} \sum_{k=n}^{rn} \frac{1}{k}$$

Thus we have only to chose r so that

$$\frac{c}{M} \log r > 1. \tag{5.11}$$

and it follows that $R(\beta)$ holds.

Next, suppose that $Q(\beta)$ holds. It is slightly more convenient to take $Q(\beta)$ in the equivalent form that there is some constant M such that, for $k \geqslant 1$,

$$\sum_{n=1}^{k} \frac{P_n}{n^{1+\beta}} \leqslant M \frac{P_k}{k^\beta}.$$

Again, we first show that (5.10) holds. As before, we may suppose $k > 2n$. Then

$$\frac{P_k}{k^\beta} \geqslant \frac{1}{M} \left\{ n^\beta \sum_{\nu=n}^{k} \frac{1}{\nu^{1+\beta}} \right\} \frac{P_n}{n^\beta}.$$

For $n \geqslant 1, k > 2n$, the expression in curly brackets is again greater than some positive constant, which again gives (5.10).

We now deduce that

$$P_{rn}/(rn)^\beta \geqslant (1/M) \sum_{k=n}^{rn} P_k/k^{1+\beta} \geqslant (cn^{-\beta} P_n/M) \sum_{k=n}^{rn} k^{-1}.$$

Thus again $R(\beta)$ follows if we choose r so that (5.11) holds.

Now, we consider the converse implications. We show that, if $R(\beta)$ holds, then there is some $\delta > 0$ such that, uniformly in $1 \leqslant v \leqslant n$,

$$P_v/P_n = O\{(v/n)^{\beta+\delta}\}. \tag{5.12}$$

It is trivial that (5.12) implies that $P(\beta)$ and $Q(\beta)$ hold, so this will give the conclusion. We remark also that this will show that (5.12) gives us another equivalent statement of the condition.

Let r be as in the definition of $R(\beta)$, and suppose that $1 \leqslant v \leqslant n$. Choose an integer s so that

$$v\, r^s \leqslant n < v\, r^{s+1}.$$

Then

$$P_n \geqslant P_{vr^s} \geqslant r^{s\beta} \lambda^s P_v = c(r^\beta \lambda)^{s+1} P_v, \tag{5.13}$$

where $c = 1/(r^\beta \lambda)$ is a positive constant. But

$$r^{s+1} > n/v$$

so that

$$\lambda^{s+1} > (n/v)^\delta,$$

where

$$\delta = \frac{\log \lambda}{\log r},$$

and is thus a positive constant. Hence (5.13) gives $P_n \geqslant c(n/v)^\delta P_v$, which gives the conclusion.

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تطبيق المصفوفة المطلقة القابلية الجمع في متسلسلات فوريه والمتسلسلات المتحالفة

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خلاصة

لقد تمت دراسة تطبيقات المتسلسلات مطلقة المحافظة على تحويل متسلسلة إلى متسلسلات فوريه والمتسلسلات المرافقة ذات العوامل . وامكن بالنتيجة تحسين نتائج سابقة كما أمكن الحصول على نتائج جديدة .