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ON CONVERGENCE IN THE ULTRA-REAL SPACE

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Abstract. The purpose of this paper is to study convergence in the Ultra-real Space *R with respect to a suitable topology Specifically we will investigate the relationship between the usual convergence in R of a real sequence and the convergence in *R of its extension (in some way) to *R.

1. PRELIMINARIES

The definitions and notations used in this work are in general those of Bourbaki (1955) and Robinson (1970).

Let u be a fixed ultra-filter on N containing the Frechet filter. Let $\mathcal{S}$ be the real algebra of all real sequences, addition, multiplication, and multiplication by scaler being defined by:

$$(f+g)_p = f_p + g_p \ ;$$
$$(f \cdot g)_p = f_p \cdot g_p \ ;$$
$$(\lambda f)_p = \lambda \cdot f_p \ .$$

The relation " $\sim$ " on $\mathcal{S}$ defined by $f \sim g$ if there exists $U \in u$ such that $f_U = g_U$, is an equivalence relation on $\mathcal{S}$. The equivalence class of $f \in \mathcal{S}$ will be denoted by *f. It is easy to verify that $\sim$ is compatible with each operation on $\mathcal{S}$. Denote by θ the equivalence class of the O-sequence. θ is two-sided ideal of $\mathcal{S}$ and

$$f \sim g \iff f - g \in \theta$$

By extending, in the usual way, the operations on $\mathcal{S}$ to the quotient $\mathcal{S}/\theta$, we obtain an algebra (Bourbaki 1955, p. 104) which is in fact a field. This algebra will be denoted by * R. The order relation $\leqslant$ on $\mathcal{S}$ defined by

$$f \leqslant g \iff \forall n \in N: \quad f_n \leqslant g_n$$

can be extended as usual to *R. *R, ordered by this extended order relation, is an ordered algebra (ordered field of course). It is called the ultra-real space. Denote by φ the mapping $a \rightarrow (a, a, \ldots)$ from R to $\mathcal{S}$, and by π the canonical mapping from $\mathcal{S}$ to * R. It is easy to see that $\pi \circ \varphi$ is an isomorphism between R , as an ordered field, and $\pi \circ \varphi$ (R). So we can identify R with the subfield $\pi \circ \varphi$ (R) of * R.

Now let $\mathcal{N}$ be the subset of $\mathcal{S}$ defined by

$$n \in \mathcal{N} \iff \forall p \in N : \ n_p \in N$$

π ($\mathcal{N}$) will be denoted by * N and its elements will be called ultra-naturals.

2. *R AS A GENERALISED NORMED SPACE

Let *f $\in$ *R, f $\in$ *f. It is easy to show that * f is independent of our choice of f $\in$ * f. So $|{}^*f| = {}^*|f|$ is well defined. The mapping *f $\rightarrow$ *f from * R into * R has the following properties:

(I) $|{}^*f| \geqslant \Theta$. and $|{}^*f| = \Theta \iff {}^*f = \Theta$;

(II) $|{}^*f + {}^*g| \leqslant |{}^*f| + |{}^*g|$:

(III) $\forall \lambda \in R : |\lambda \, {}^*f| = |\lambda| . |{}^*f|$.

These properties, being similar to those of a norm, it is natural to call $|{}^*f|$ an ultra-norm.

An element *f $\in$ * R will be called inite-simal if $|{}^*f| \leqslant a$, for all $a > O$ with $\in$ R. The set of all infinitesimals will be denoted by $\mathcal{T}$. *f is called finite if there exists a $\in$ R such that $|{}^*f| \leqslant a$. *f is called infinite if it is not finite. We note that $|{}^*f|$ defines a uniformity on * R. We shall consider always this uniformity and the topology induced by it.

3. The Extended Sequences

If $f \in \mathcal{J}$, $^*n \in {}^*N$; $n, m \in {}^*n$, we can verify that $f \circ n$, $f \circ m$ are defined in some set $U \in \mathcal{U}$. If we define arbitrarily $f \circ n$ and $f \circ m$ outside U, we have $f \circ n, f \circ m \in \mathcal{J}$ and $f \circ n \sim f \circ m$, so that $^*(f \circ n) = {}^*(f \circ m)$. If we denote $^*(f \circ n)$ by f_{*n} then the mapping $F: {}^*n \to f_{*n}$ from $^*N$ into $^*R$ is said to be the extended sequence of f. It is shown (Robinson 1970) by logical methods that f converges in the usual sense in R iff for all infinite ultra-naturals $^*n, {}^*m$ we have $f_{*n} - f_{*m} \in \mathscr{T}$. We prove directly the following:

Theorem

Let f be a real sequence. Then the following four statements are equivalent:

(a) F is a Cauchy extended sequence;

(b) $f_{*n} - f_{*m} \in \mathscr{T}$ for all infinite ultra-naturals $^*n, {}^*m$;

(c) f is convergent in R (in the usual sense);

(d) F is convergent in $^*R$.

First we will prove the following lemma.

Lemma

If f is a real sequence and $^*n, {}^*m$ are infinite ultra-naturals such that $f_{*n} - f_{*m} \notin \mathscr{T}$ then for any $^*n_o \in {}^*N$, there exists ultra-naturals $^*n', {}^*m' > {}^*n_o$ such that $f_{*n'} - f_{*m'} \notin \mathscr{T}$.

Proof:

Let $^*n, {}^*m$ be infinite ultra-naturals such that $|f_{*n} - f_{*m}| > \varepsilon$ where ε is a positive standard real. Let $n \in {}^*n$, $m \in {}^*m$ and $n^o \in {}^*n_o$. Then there exists some $U \in \mu$ such that:

$$i \in U \Rightarrow n_i, m_i, n_i^o \in N \text{ and } |f_{n_i} - f_{m_i}| > \varepsilon .$$

Since $^*n$ and $^*m$ are infinite ultra-naturals, then we can choose $k' \in U$ for any $k \in U$ such that $n_{k'} > \max\{n_k, n_k^o\}$ and $m_{k'} > \max\{m_k, n_k^o\}$.

Then we have obviously $|f_{n_{k'}} - f_{m_{k'}}| > \varepsilon$.

Define n', m' by $n'_k = n_{k'}$, $m'_k = m_{k'}$ on U and arbitrarily on U^c. Now if $^*n'$, $^*m'$ are defined by n', m' respectively, then we have:

$$^*n', {}^*m' > {}^*n_o \text{ and } |f_{*n'} - f_{*m'}| > \varepsilon$$

Now we present the proof of the theorem.

(a) $\Rightarrow$ (b)

If F is a Cauchy extended sequence, then for the infinitesimal $^*\mu > \theta$, $\exists {}^*n_o$ such that:

$$^*n, {}^*m > {}^*n_o \Rightarrow |f_{*n} - f_{*m}| < {}^*\mu .$$

Now suppose $^*n, {}^*m$ are infinite ultra-naturals such that $|f_{*n} - f_{*m}| > \varepsilon$ with ε a positive standard real. Hence by the lemma, there exists $^*n', {}^*m' > {}^*n_o$ such that $f_{*n} - f_{*m} \notin \mathscr{T}$; a contradiction.

(b) $\Longrightarrow$ (c)

Suppose to the contrary that f is not convergent then f is not a Cauchy sequence (in the usual sense). Hence $\exists\ \varepsilon > 0$ (with $\varepsilon \in R$): $\forall p \in N, \exists n, m > p$ such that $|f_n - f_m| > \varepsilon$.

This means that there exist two infinite increasing sequences of real naturals, say (n_p) and (m_p), such that $|f_{n_p} - f_{m_p}| > \varepsilon$ for all n_p and m_p. Let $^*n, {}^*m$ be defined by (n_p) and (m_p) respectively. Then $^*n, {}^*m$ are obviously infinite ultra-naturals and $|f_{*n} - f_{*m}| > \varepsilon$; a contradiction.

(c) $\Longrightarrow$ (d)

Let f be convergent to $d \in R$ and $^*\mu$ be an arbitrary positive infinitesimal. Let $\mu \in {}^*\mu$ with $\mu_k > 0$ for all $k \in N$. Now for all k, $\exists\ n_k$: $p > n_k \Longrightarrow |f_p - d| < \mu_k$. Let $^*n_o$ be defined by (n_k). Then obviously $|f_{*n} - d| < {}^*\mu$ for all $^*n > {}^*n_o$.

(d) $\Longrightarrow$ (a)

Obvious.

Remarks

(1) This theorem shows that f and F converge or diverge together. Moreover, they have the same limit in the case of convergence.

(2) Though *R does not have the Dedekind property, it does have the Cauchy completeness property in the sense of this theorem.

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حول التقارب في الفضاء فوق الحقيقي

صلاح احمد وعادل ياسين

قسم الرياضيــات بجامعة الكويت

خلاصـــة

في هذا البحث تمت دراسة التقارب في الفضاء فوق الحقيقي ح* تبعا لطبولوجيا مناسبة . وأمكن التوصل ، بوجه خاص ، الى العلاقة ما بين التقارب الاعتيادي لمتتالية حقيقية في ح والتقارب لامتدادها (بطريقة ما) في ح*.