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Contents

MATHEMATICS

CHEMISTRY

BOTANY

MICROBIOLOGY

ZOOLOGY

GEOLOGY

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The multicolourings of graphs and hypergraphs

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ABSTRACT

In this paper, conditions for the existence of multicolouring and for the existence of a stable set in a hypergraph are provided.

1. INTRODUCTION

Let $H = (X, \xi)$ be a hypergraph with vertex-set $X = \{x_1, x_2, \ldots, x_n\}$. We have a *multicolouring* of H with λ colours $C_1, C_2, \ldots, C_\lambda$ if we can assign to each $x \in X$ one or several of these colours so that in each edge of H every colour occurs and occurs exactly once.

Clearly, if there exists a multicolouring of H, one colour defines a set of vertices which is both strongly stable and transversal; no general condition for the existence of such a set exists in the literature. The purpose of this paper is to provide existence conditions, not only for a stable transversal set, but also for a multicolouring.

This problem arises for a multicolouring of the edges of a graph G, i.e. a multicolouring of the vertices of the dual G^* of G. If a regular graph G has a chromatic index $q(G)$ equal to the maximum degree $\Delta(G)$, then there exists a multicolouring of its edges; the Petersen graph P_{10} does not satisfy $q(P_{10}) = \Delta(P_{10})$, but nevertheless one can find a multicolouring of its edges. Thus, the existence of a multicolouring can be considered as an extension of the edge colouring property.

Another context for this problem is the linear programming. Clearly, a multicolouring is given by the solution of a system of linear inequalities in integers, and all the classical tools can be used. When the hypergraph H is balanced, the linear programming formulation is still simpler, as one can see from the theorem 4 of this paper (which is also a new generalization of the Birkhoff theorem on bi-stochastic matrices).

2. MULTICOLOURINGS FOR INTERVAL HYPERGRAPHS AND BIPARTITE GRAPHS

The first hypergraphs we shall consider are the interval hypergraphs: the vertices represent points of the line, and the edges represent intervals.

Theorem 1. An interval hypergraph has no multicolouring if and only if there exist two edges A and B such that

$$A \subset B, \quad A \neq B.$$

(In other words, an interval hypergraph is multicolourable iff it is a 'Sperner hypergraph'.)

1. If there exist two edges A and B such that $A \subset\subset B$, then a multicolouring with λ colours is impossible, because A would contain λ colours, and therefore B would contain more than λ colours, which is a contradiction.

2. Let $H = (A_i/i \in M)$ be an interval hypergraph satisfying

$$A_i \subset A_j \Rightarrow i = j$$

We shall show that there exists a multicolouring of H. It suffices to show that every vertex x_0 is contained in a set S which is both strongly stable and transversal.

Let $H(x_0)$ be defined by the intervals which contain x_0. Let $H^+(x_0)$ be defined by the intervals having their initial endpoint at the right side of x_0. Let $H^-(x_0)$ be defined by the intervals having their terminal endpoint at the left side of x_0 on the line. Thus,

$$H(x_0) + H^+(x_0) + H^-(x_0) = H$$

Let A_0 be the (unique) interval of $H(x_0)$ with a terminal endpoint as far as possible from x_0. Let x_1 be the first vertex at the right-hand side of A, and let A_1 be the interval of $H(x_1)$ with a terminal endpoint as far as possible from x_1. We can similarly define a vertex x_2, etc. ... The set $S^+ = \{x_0, x_1, x_2, \ldots\}$ is both transversal and strongly stable for the union $H(x_0) + H^+(x_0)$. By the same method, we can define a set $S^- = \{x_0, x_{-1}, x_{-2}, \ldots\}$ which is transversal and strongly stable for $H(x_0) + H^-(x_0)$. The union $S^+ \bigcap S^-$ is transversal and strongly stable for H.

Q.E.D.

Let $d_H(x)$ denote the *degree* of a vertex x of H, i.e. the number of edges of H containing x. Here, we may have $d_H(x) = 0$ (if x is an isolated vertex).

Theorem 2. Let H be a hypergraph. If there exist two partial hypergraphs H' and H'' with the same number of edges, such that

(i) $\quad d_{H'}(x) \leqslant d_{H''}(x) \quad (x \in X)$

(ii) $\quad d_{H'}(x) < d_{H''}(x) \quad$ for some $x \in X$,

then H has no multicolouring.

Let $H' = (E_i/i \in I)$ and $H'' = (E_j/j \in J)$. So, $|I| = |J|$, and we may assume that $I \cap J = \varnothing$ (otherwise, remove the common edges from H' and H'').

Put

$$z_i = \begin{cases} +1 & \text{if } i \in J \\ -1 & \text{if } i \in I \\ 0 & \text{if } i \notin I \bigcup J \end{cases}$$

Denote by $\phi_i(x)$ the characteristic function of the set E_i.

Suppose that theorem 2 is false: there exists a multicolouring with λ colours. Let

$\lambda(x)$ be the number of colours assigned to the vertex x in this multicolouring. We have

$$\begin{aligned} 0 &< \sum_x \lambda(x)[d_{H''}(x) - d_{H'}(x)] = \\ &= \sum_x \lambda(x)[\sum_{i\in J} \phi_i(x) - \sum_{i\in I} \phi_i(x)] = \\ &= \sum_x \lambda(x) \sum_{i\in M} z_i \phi_i(x) = \sum_{i\in M} z_i \sum_x \phi_i(x)\lambda(x) \\ &= \sum_{i\in M} z_i \lambda = (|J| - |I|)\lambda = 0. \end{aligned}$$

The contradiction follows.

Q.E.D.

Theorem 3. Let $H = (E_i/i \in M)$ be the dual of a bipartite graph G. Then H has no multicolouring if and only if there exist two sets $I, J \subset M$ satisfying:

$$(1) \qquad \begin{cases} |I| = |J| & \\ E_i \cap E_{i'} = \varnothing & (i, i' \in I,\ i \neq i') \\ E_j \cap E_{j'} = \varnothing & (j, j' \in J,\ j \neq j') \\ \bigcup_{i\in I} E_i \subset\subset \bigcup_{j\in J} E_j. & \end{cases}$$

Proof. If there exist two sets I and J satisfying (1), then from theorem 2, H has no multicolouring.

Conversely, let $G = (X', X'', \Gamma)$ be a bipartite graph with no multicolouring of its edges. Put $X' = \{x_i/i \in P\}$, $X'' = \{x_j/j \in Q\}$, and denote by E_i the set of edges incident to x_i. So, $H = (E_i/i \in P \cup Q)$.

Case 1. $|P| \neq |Q|$.

Assume $|P| < |Q|$. Take $I = P$, and let J be a subset of Q with cardinality $|P|$. Then I and J satisfy (1), and the proof is achieved.

Case 2. $|P| = |Q|$.

Then there exist two adjacent vertices $a' \in X'$ and $a'' \in X''$ such that the subgraph $\bar{G}$ induced by $(X' - \{a'\}) \cup (X'' - \{a''\})$ has no perfect matching.

By the König theorem, there exists a set $A \subset X'$ such that

$$|A| > |A - \{a''\}|$$

If $|A| = |\Gamma A|$, then $a'' \in \Gamma A$. The sets $I = A$ and $J = \Gamma A$ satisfy (1) and the proof is achieved.

If $|A| > |\Gamma A|$, take $I \subset A$ with cardinality $|\Gamma A|$ and $J = \Gamma A$; these sets satisfy (1), and the proof is achieved.

Remark. This theorem can be extended to the dual of a graph $G = (X, E)$ satisfying

(i) X is even

(ii) no two odd cycles are disjoint and non-adjacent.

This class of graphs, called semi-bipartite, includes the Petersen graph. The dual of a semi-bipartite graph G has no multicolouring if and only if the condition of theorem 2 is fulfilled. This follows from theorem 8 in (Berge, 1973, chap. 7).

3. MULTICOLOURINGS FOR BALANCED HYPERGRAPHS

In this section, we shall investigate the existence of multicolourings for a more general class of hypergraphs that we introduced in a previous paper (Berge 1970) to generalize the unimodular matrices:

A hypergraph H is said to be *balanced* if every odd cycle $(x_1, E_1, x_2, E_2, \ldots, x_{2k+1}, E_{2k+1}, x_1)$ contains an edge E_i which contains three of the x_i's.

Let $H = (X, \xi)$ be a hypergraph on X; consider a function $\phi: \mathrm{X} \to \mathrm{R}$ such that

$$\phi(x) > 0 \qquad (x \in X)$$
$$\sum_{x \in E_i} \phi(x) = 1 \qquad (i \in M)$$

Following Csima (1970), such a function will be called a positive stochastic function of H. The following result generalizes Theorem 10, chap. 20, in Berge (1973):

Theorem 4. If H is multicolourable, then H has a positive stochastic function; furthermore, when H is balanced, the converse is also true.

Proof. If H has a multicolouring with λ colours, let $\lambda(x)$ be the number of colours assigned to the vertex x. Clearly, $\lambda^{-1}\lambda(x)$ is a positive stochastic function for H.

Conversely, let H be a balanced hypergraph with a stochastic function $\phi(x)$. Since the vector $(\phi(x_1), \phi(x_2), \ldots, \phi(x_n))$ is a solution of a system of linear inequalities with integer coefficients, we may assume that all the $\phi(x_i)$s are rational numbers, and we write for all i:

$$\phi(x_i) = \frac{p_i}{p}$$

where p_i and p are positive integers.

Let $H' = (X', (E_i'))$ be the hypergraph obtained from H by duplicating p_i times the vertex x_i, that is:

$$X' = X_1 \cup X_2 \cup \ldots \cup X_n$$
$$|X_i| = p_i$$
$$X_i \cap X_j = \phi \qquad (i \neq j)$$

and

$$E_j' = \bigcup \{X_i / x_i \in E_j\}$$

Clearly, the hypergraph H' is also balanced. Furthermore, H' is uniform, because for all j,

$$|E_j'| = \sum_{x_i \in E_j} |X_i| = \sum_{x_i \in E_j} p_i = p \sum_{x \in E_j} \phi(x) = p$$

From the corollary to theorem 6 (Berge 1973), chap. 20), it is possible to colour the vertices of H' with p colours such that each edge E_i has all its vertices with different colours. This colouring defines also a multicolouring of H with p colours.

Q.E.D.

Remark. If H is not balanced, it is not necessarily true that the existence of a positive stochastic function yields the existence of a multicolouring. The graph K_3 has a stochastic function (with value 1/2 at each vertex), but no multicolouring.

We can easily see that theorem 4 is a generalization of the Birkhoff–Von Neumann theorem on bistochastic matrices: 'Every bistochastic matrix is a barycentre of permutation matrices'.

Let A be a $n \times n$ bistochastic matrix. We shall show that $A = \Sigma\, \lambda_i P_i$, where the P_i' are permutation matrices, $\lambda_i > 0$ and $\Sigma\, \lambda_i = 1$. Let H be a hypergraph whose vertices are the positions of the non-zero entries of A, an edge of H being defined by all the positions located on a same column or on a same row. This hypergraph is balanced (because H has no odd cycles) and possesses a stochastic function (defined by the entries of the matrix A). By theorem 3, H has a multicolouring. Let λ_1 be the least positive coefficient of A, and let x be the corresponding vertex of H. Let α_1 be the colour assigned to x. The (0, 1)-matrix P_1, with a 1 in each position corresponding to a vertex of H with colour α_1, is a permutation matrix; $A' = (A - \lambda_1 P_1)(1-\lambda)^{-1}$ is a bistochastic matrix having more zero entries than A. If we repeat the same reduction with A', as many times as needed, we get the required decomposition.

Q.E.D.

Theorem 5. Let $H = (X, \xi)$ be a hypergraph on X; for $E_i \in \xi$, let $\phi_i(x)$ be the characteristic function of E_i. If there exist integers $z_1, z_2, \ldots, z_m \in Z$ such that:

$$(1) \qquad \begin{cases} \sum_{i=1}^{m} z_i = 0 & \\ \sum_{i=1}^{n} z_i \phi_i(x) \geqslant 0 & (x \in X) \\ \sum z_i \phi_i(x) \neq 0 & \text{for some } x. \end{cases}$$

then H is not multicolourable.

Furthermore, for a balanced hypergraph, the converse is also true.

Proof. The existence of integers z_i (positive, negative or null) satisfying (1) implies that H is not multicolourable: the proof is exactly as in theorem 2. Now, let H be a balanced hypergraph which is not multicolourable. From theorem 4, it follows that H has no positive stochastic function. Then, from the theorem of Csima (1970), the existence of z_is satisfying (1) follows.

Q.E.D.

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