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The multicolourings of graphs and hypergraphs

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ABSTRACT

In this paper, conditions for the existence of multicolouring and for the existence of a stable set in a hypergraph are provided.

1. INTRODUCTION

Let $H = (X, \xi)$ be a hypergraph with vertex-set $X = \{x_1, x_2, \dots, x_n\}$. We have a *multicolouring* of H with λ colours $C_1, C_2, \dots, C_\lambda$ if we can assign to each $x \in X$ one or several of these colours so that in each edge of H every colour occurs and occurs exactly once.

Clearly, if there exists a multicolouring of H , one colour defines a set of vertices which is both strongly stable and transversal; no general condition for the existence of such a set exists in the literature. The purpose of this paper is to provide existence conditions, not only for a stable transversal set, but also for a multicolouring.

This problem arises for a multicolouring of the edges of a graph G , i.e. a multicolouring of the vertices of the dual G^* of G . If a regular graph G has a chromatic index $q(G)$ equal to the maximum degree $\Delta(G)$, then there exists a multicolouring of its edges; the Petersen graph P_{10} does not satisfy $q(P_{10}) = \Delta(P_{10})$, but nevertheless one can find a multicolouring of its edges. Thus, the existence of a multicolouring can be considered as an extension of the edge colouring property.

Another context for this problem is the linear programming. Clearly, a multicolouring is given by the solution of a system of linear inequalities in integers, and all the classical tools can be used. When the hypergraph H is balanced, the linear programming formulation is still simpler, as one can see from the theorem 4 of this paper (which is also a new generalization of the Birkhoff theorem on bi-stochastic matrices).

2. MULTICOLOURINGS FOR INTERVAL HYPERGRAPHS AND BIPARTITE GRAPHS

The first hypergraphs we shall consider are the interval hypergraphs: the vertices represent points of the line, and the edges represent intervals.

Theorem 1. An interval hypergraph has no multicolouring if and only if there exist two edges A and B such that

$$A \subset B, \quad A \neq B.$$

(In other words, an interval hypergraph is multicolourable iff it is a ‘Sperner hypergraph’.)

1. If there exist two edges A and B such that $A \subset \subset B$, then a multicolouring with λ colours is impossible, because A would contain λ colours, and therefore B would contain more than λ colours, which is a contradiction.

2. Let $H = (A_i/i \in M)$ be an interval hypergraph satisfying

$$A_i \subset A_j \Rightarrow i = j$$

We shall show that there exists a multicolouring of H . It suffices to show that every vertex x_0 is contained in a set S which is both strongly stable and transversal.

Let $H(x_0)$ be defined by the intervals which contain x_0 . Let $H^+(x_0)$ be defined by the intervals having their initial endpoint at the right side of x_0 . Let $H^-(x_0)$ be defined by the intervals having their terminal endpoint at the left side of x_0 on the line. Thus,

$$H(x_0) + H^+(x_0) + H^-(x_0) = H$$

Let A_0 be the (unique) interval of $H(x_0)$ with a terminal endpoint as far as possible from x_0 . Let x_1 be the first vertex at the right-hand side of A_0 , and let A_1 be the interval of $H(x_1)$ with a terminal endpoint as far as possible from x_1 . We can similarly define a vertex x_2 , etc. . . . The set $S^+ = \{x_0, x_1, x_2, \dots\}$ is both transversal and strongly stable for the union $H(x_0) + H^+(x_0)$. By the same method, we can define a set $S^- = \{x_0, x_{-1}, x_{-2}, \dots\}$ which is transversal and strongly stable for $H(x_0) + H^-(x_0)$. The union $S^+ \cap S^-$ is transversal and strongly stable for H .

Q.E.D.

Let $d_H(x)$ denote the *degree* of a vertex x of H , i.e. the number of edges of H containing x . Here, we may have $d_H(x) = 0$ (if x is an isolated vertex).

Theorem 2. Let H be a hypergraph. If there exist two partial hypergraphs H' and H'' with the same number of edges, such that

- (i) $d_{H'}(x) \leq d_{H''}(x) \quad (x \in X)$
- (ii) $d_{H'}(x) < d_{H''}(x) \quad \text{for some } x \in X,$

then H has no multicolouring.

Let $H' = (E_i/i \in I)$ and $H'' = (E_j/j \in J)$. So, $|I| = |J|$, and we may assume that $I \cap J = \emptyset$ (otherwise, remove the common edges from H' and H'').

$$\text{Put} \quad \begin{aligned} z_i &= +1 && \text{if } i \in J \\ &= -1 && \text{if } i \in I \\ &= 0 && \text{if } i \notin I \cup J \end{aligned}$$

Denote by $\phi_i(x)$ the characteristic function of the set E_i .

Suppose that theorem 2 is false: there exists a multicolouring with λ colours. Let

$\lambda(x)$ be the number of colours assigned to the vertex x in this multicolouring. We have

$$\begin{aligned} 0 &< \sum_x \lambda(x) [d_{I''}(x) - d_{J''}(x)] = \\ &= \sum_x \lambda(x) \left[\sum_{i \in J} \phi_i(x) - \sum_{i \in I} \phi_i(x) \right] = \\ &= \sum_x \lambda(x) \sum_{i \in M} z_i \phi_i(x) = \sum_{i \in M} z_i \sum_x \phi_i(x) \lambda(x) \\ &= \sum_{i \in M} z_i \lambda = (|J| - |I|) \lambda = 0. \end{aligned}$$

The contradiction follows.

Q.E.D.

Theorem 3. Let $H = (E_i | i \in M)$ be the dual of a bipartite graph G . Then H has no multicolouring if and only if there exist two sets $I, J \subset M$ satisfying:

$$(1) \quad \begin{cases} |I| = |J| \\ E_i \cap E_{i'} = \emptyset & (i, i' \in I, i \neq i') \\ E_j \cap E_{j'} = \emptyset & (j, j' \in J, j \neq j') \\ \bigcup_{i \in I} E_i \subset \bigcup_{j \in J} E_j. \end{cases}$$

Proof. If there exist two sets I and J satisfying (1), then from theorem 2, H has no multicolouring.

Conversely, let $G = (X', X'', \Gamma)$ be a bipartite graph with no multicolouring of its edges. Put $X' = \{x_i | i \in P\}$, $X'' = \{x_j | j \in Q\}$, and denote by E_i the set of edges incident to x_i . So, $H = (E_i | i \in P \cup Q)$.

Case 1. $|P| \neq |Q|$.

Assume $|P| < |Q|$. Take $I = P$, and let J be a subset of Q with cardinality $|P|$. Then I and J satisfy (1), and the proof is achieved.

Case 2. $|P| = |Q|$.

Then there exist two adjacent vertices $a' \in X'$ and $a'' \in X''$ such that the subgraph \bar{G} induced by $(X' - \{a'\}) \cup (X'' - \{a''\})$ has no perfect matching.

By the König theorem, there exists a set $A \subset X'$ such that

$$|A| > |A - \{a''\}|$$

If $|A| = |\Gamma A|$, then $a'' \in \Gamma A$. The sets $I = A$ and $J = \Gamma A$ satisfy (1) and the proof is achieved.

If $|A| > |\Gamma A|$, take $I \subset A$ with cardinality $|\Gamma A|$ and $J = \Gamma A$; these sets satisfy (1), and the proof is achieved.

Remark. This theorem can be extended to the dual of a graph $G = (X, E)$ satisfying

- (i) X is even
- (ii) no two odd cycles are disjoint and non-adjacent.

This class of graphs, called semi-bipartite, includes the Petersen graph. The dual of a semi-bipartite graph G has no multicolouring if and only if the condition of theorem 2 is fulfilled. This follows from theorem 8 in (Berge, 1973, chap. 7).

3. MULTICOLOURINGS FOR BALANCED HYPERGRAPHS

In this section, we shall investigate the existence of multicolourings for a more general class of hypergraphs that we introduced in a previous paper (Berge 1970) to generalize the unimodular matrices:

A hypergraph H is said to be *balanced* if every odd cycle $(x_1, E_1, x_2, E_2, \dots, x_{2k+1}, E_{2k+1}, x_1)$ contains an edge E_i which contains three of the x_i 's.

Let $H = (X, \xi)$ be a hypergraph on X ; consider a function $\phi: X \rightarrow \mathbf{R}$ such that

$$\begin{aligned} \phi(x) &> 0 & (x \in X) \\ \sum_{x \in E_i} \phi(x) &= 1 & (i \in M) \end{aligned}$$

Following Csima (1970), such a function will be called a positive stochastic function of H . The following result generalizes Theorem 10, chap. 20, in Berge (1973):

Theorem 4. If H is multicolourable, then H has a positive stochastic function; furthermore, when H is balanced, the converse is also true.

Proof. If H has a multicolouring with λ colours, let $\lambda(x)$ be the number of colours assigned to the vertex x . Clearly, $\lambda^{-1}\lambda(x)$ is a positive stochastic function for H .

Conversely, let H be a balanced hypergraph with a stochastic function $\phi(x)$. Since the vector $(\phi(x_1), \phi(x_2), \dots, \phi(x_n))$ is a solution of a system of linear inequalities with integer coefficients, we may assume that all the $\phi(x_i)$ s are rational numbers, and we write for all i :

$$\phi(x_i) = \frac{p_i}{p}$$

where p_i and p are positive integers.

Let $H' = (X', (E'_i))$ be the hypergraph obtained from H by duplicating p_i times the vertex x_i , that is:

$$\begin{aligned} X' &= X_1 \cup X_2 \cup \dots \cup X_n \\ |X_i| &= p_i \\ X_i \cap X_j &= \emptyset \quad (i \neq j) \end{aligned}$$

and

$$E'_j = \bigcup \{X_i / x_i \in E_j\}$$

Clearly, the hypergraph H' is also balanced. Furthermore, H' is uniform, because for all j ,

$$|E'_j| = \sum_{x_i \in E_j} |X_i| = \sum_{x_i \in E_j} p_i = p \sum_{x \in E_j} \phi(x) = p$$

From the corollary to theorem 6 (Berge 1973), chap. 20), it is possible to colour the vertices of H' with p colours such that each edge E'_j has all its vertices with different colours. This colouring defines also a multicolouring of H with p colours.

Q.E.D.

Remark. If H is not balanced, it is not necessarily true that the existence of a positive stochastic function yields the existence of a multicolouring. The graph K_3 has a stochastic function (with value $1/2$ at each vertex), but no multicolouring.

We can easily see that theorem 4 is a generalization of the Birkhoff–Von Neumann

theorem on bistochastic matrices: 'Every bistochastic matrix is a barycentre of permutation matrices'.

Let A be a $n \times n$ bistochastic matrix. We shall show that $A = \sum \lambda_i P_i$, where the P_i are permutation matrices, $\lambda_i > 0$ and $\sum \lambda_i = 1$. Let H be a hypergraph whose vertices are the positions of the non-zero entries of A , an edge of H being defined by all the positions located on a same column or on a same row. This hypergraph is balanced (because H has no odd cycles) and possesses a stochastic function (defined by the entries of the matrix A). By theorem 3, H has a multicolouring. Let α_1 be the least positive coefficient of A , and let x be the corresponding vertex of H . Let α_1 be the colour assigned to x . The $(0, 1)$ -matrix P_1 , with a 1 in each position corresponding to a vertex of H with colour α_1 , is a permutation matrix; $A' = (A - \lambda_1 P_1)(1 - \lambda_1)^{-1}$ is a bistochastic matrix having more zero entries than A . If we repeat the same reduction with A' , as many times as needed, we get the required decomposition.

Q.E.D.

Theorem 5. Let $H = (X, \xi)$ be a hypergraph on X ; for $E_i \in \xi$, let $\phi_i(x)$ be the characteristic function of E_i . If there exist integers $z_1, z_2, \dots, z_m \in \mathbb{Z}$ such that:

$$(1) \quad \begin{cases} \sum_{i=1}^m z_i = 0 \\ \sum_{i=1}^n z_i \phi_i(x) \geq 0 & (x \in X) \\ \sum z_i \phi_i(x) \neq 0 & \text{for some } x. \end{cases}$$

then H is not multicolourable.

Furthermore, for a balanced hypergraph, the converse is also true.

Proof. The existence of integers z_i (positive, negative or null) satisfying (1) implies that H is not multicolourable: the proof is exactly as in theorem 2. Now, let H be a balanced hypergraph which is not multicolourable. From theorem 4, it follows that H has no positive stochastic function. Then, from the theorem of Csima (1970), the existence of z_i s satisfying (1) follows.

Q.E.D.

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خلاصة

لقد أمكن الحصول على شروط ضرورية لوجود مجموعات مستقرة قاطعة وكذلك تلوين ما فوق البيانات .