*J. Univ. Kuwait (Sci.)* **3**, 1976

# Instability of an elastically restrained column subjected to a follower force

JASIM ALSAIGH AND PAUL CHRISTIANO

*Department of Civil Engineering, University of Kuwait, and Department of Civil Engineering, Carnegi Mellon University, Pittsburg, Pennsylvania, U.S.A.*

## ABSTRACT

Theoretical investigation of a cantilever column restrained at its tip by a rotational spring and subjected to a follower force is made. Depending on the stiffness of the spring either flutter (i.e. self-excited oscillation) or divergent (i.e. static) type instability is reported.

## 1. INTRODUCTION

Among the various types of nonconservative stability problems, that dealing with elastic systems subjected to follower forces has been the focus of great interest in the past two decades. A review of this subject up to 1967 was presented by Hermann (1967), and books by Bolotin (1963) and Ziegler (1968) are now standard references.

A study of the effect on instability of loading having conservative and nonconservative components was reported by Hermann & Bungay (1964). Using a two-degree-of-freedom model of a cantilevered column subjected to an axial load and an arbitrary tangential component, they showed that the type of instability (i.e. either divergence or flutter) depends on the ratio of the two forces. Further, when the loads are increased proportionally, multiple stable and unstable ranges of load exist. The relationship between modes of instability and the degree to which a system is nonconservative may be used to resolve an apparent paradox below.

## 2. THE ELASTICALLY RESTRAINED COLUMN SUBJECTED TO A FOLLOWER FORCE

Consider a column of length $L$, and flexural stiffness $EI$, clamped at its base and subjected to a concentrated axial force $P$, at its free end. As mentioned in many references, the critical value in the case of the follower force (Fig. 1b) is over eight times that in the case of the conservative force (Fig. 1a). Although not usually noted, the follower force causing instability of the cantilevered column is also over twice as great as the critical load (necessarily conservative) for a column clamped at the base, and fixed against rotation but free to translate at the top (Fig. 1d). Comparison of cases (b) and (d) is interesting because each may be viewed as a different limiting condition of

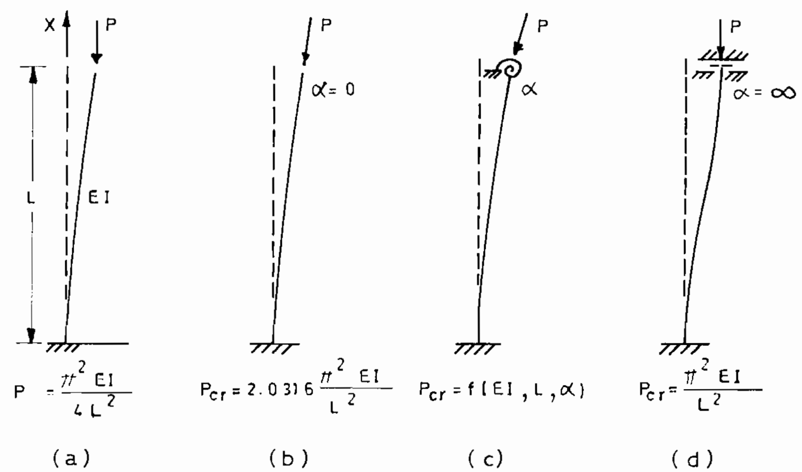


**Fig. 1.** Column under various loading and boundary conditions.

the system shown in (Fig. 1c) in which a column is clamped at the base, and elastically restrained against rotation, but otherwise free at the top.

Under the action of a follower force, the critical load approaches that of case (b) as the rotational spring constant $\alpha$, tends to zero, and approaches that of case (d) as $\alpha$ tends to infinity. The follower force causing instability is, therefore, greater in the limiting case having the lesser overall stiffness. It is the purpose of this paper to show the relationship between the value of the critical load and the magnitude of the rotational spring constant. It may be mentioned that $\alpha$ equal to 6 $EI/L$ pertains to the case of a portal frame having identical members and possessing a sidesway mode of instability.

The governing equation for determining stability of a column by the dynamic method is

$$y'^{v}(x,t)+k^2y''(x,t)+a\ddot{y}(x,t)=0 \qquad 0 \leqslant x \leqslant L \tag{1}$$

The geometric boundary conditions are

$$y(O,t)=0, y'(O,t)=0 \tag{2a,b}$$

and the dynamic boundary conditions are

$$y'(L,t)+\frac{EI}{\alpha}y''(L,t)=0, \; y'''(L,t)=0 \tag{2c,d}$$

where the prime and dot denote derivatives with respect to $x$ and time, $t$, respectively, $m$ is the mass per unit length of the column, and

$$k^2 \equiv \frac{P}{EI} \qquad \text{and} \qquad a \equiv \frac{m}{EI}$$

assuming a solution of the form

$$y(x,t)=A_0 f(x)\, e^{i\omega t} \tag{3}$$

the governing equation (1) is satisfied by setting

$$f(x) = A\cosh\lambda_1 x + B\sinh\lambda_1 x + C\cos\lambda_2 x + D\sin\lambda_2 x \tag{4}$$

where

$$\lambda_1 \equiv \left[\left(a\omega^2 + \frac{k^4}{4}\right)^{\frac{1}{2}} - \frac{k^2}{2}\right]^{\frac{1}{2}}$$

$$\lambda_2 \equiv \left[\left(a\omega^2 + \frac{k^4}{4}\right)^{\frac{1}{2}} + \frac{k^2}{2}\right]^{\frac{1}{2}}$$

Substituting equation (4) into the boundary conditions (2), and setting the determinant of the coefficients $A$, $B$, $C$, $D$, equal to zero, the following characteristic equation is obtained:

$$\begin{aligned}&(\lambda_1^3\,\mathrm{sh} - \lambda_2^3\,\mathrm{s})[\lambda_1^2\,\mathrm{sh} + \lambda_1\lambda_2\,\mathrm{s} + \beta(\lambda_1\,\mathrm{ch} - \lambda_1\,\mathrm{c})]\\ &-(\lambda_1^3\,\mathrm{ch} + \lambda_1\lambda_2^2\,\mathrm{c})[\lambda_1^2\,\mathrm{ch} + \lambda_2^2\,\mathrm{c} + \beta(\lambda_1\,\mathrm{sh} + \lambda_2\,\mathrm{s})] = 0\end{aligned} \tag{5}$$

in which

$$\mathrm{ch} = \cosh\lambda_1 L \qquad \mathrm{sh} = \sinh\lambda_1 L$$
$$\mathrm{c} = \cos\lambda_2 L \qquad \mathrm{s} = \sin\lambda_2 L$$

$$\beta = \frac{\alpha L}{EI}$$

Hermann & Bungay (1964) showed that for a two-degree-of-freedom model of a cantilevered column the frequency equation is quadratic in $\omega^2$, and the frequency curves ($P$ versus $\omega^2$) are of the hyperbolic type. For each of the two branches of one type of hyperbola, a single real value of $\omega^2$ occurs under any load and, therefore, only divergent instability is possible. However, in the second type of hyperbola, a given load yields two real values of $\omega^2$ which lie on the same branch of the curve. The critical load, at which the two values of $\omega^2$ coincide, implies that the frequency equation (5) has a double root. With a further increase of $P$, the roots become complex. This implies, according to equation (3), that $y$ increases indefinitely with time, i.e. self-excited oscillation indicating that the column is unstable. This is a flutter type instability.

Although the relationship between the roots of equation (5) and the applied load is more complicated than that associated with the discrete parameter model, the general shapes of the $P$ versus $\omega^2$ curves are similar to the hyperbolic forms. As shown in Fig. 2, the frequency curve for the case with a free end ($\beta = 0$) is similar to the hyperbola of the second type described above. The first two natural frequencies coincide for a value of $\bar{P} = 2{\cdot}031$ ($\bar{P} = PL^2/\pi^2 EI$), which is in agreement with that obtained by Huang *et al.* (1967). As the relative spring stiffness $\beta$, increases, the frequency curve appears to tilt until, at a value of $\beta = 4{\cdot}603$, one point touches the axis $\bar{\omega}^2 = 0$ ($\bar{\omega}^2 = aL^4/\pi^4\,\omega^2$) at $\bar{P} = 2{\cdot}028$. At this load divergent instability (i.e. buckling) occurs. Above this load, however, the column is stable until flutter instability is produced at $\bar{P} = 2{\cdot}946$. For the special case of the portal frame ($\beta = 6{\cdot}0$), divergent instability occurs at $\bar{P} = 1{\cdot}491$, and the structure remains unstable until $\bar{P} = 2{\cdot}767$. Between $\bar{P} = 2{\cdot}767$ and $3{\cdot}064$ stability is maintained; at the upper value flutter instability occurs. At a value of $\beta = 12{\cdot}0$, the load causing divergent instability is $\bar{P} = 1{\cdot}195$, and there is no higher load at which stability is regained, that is, for

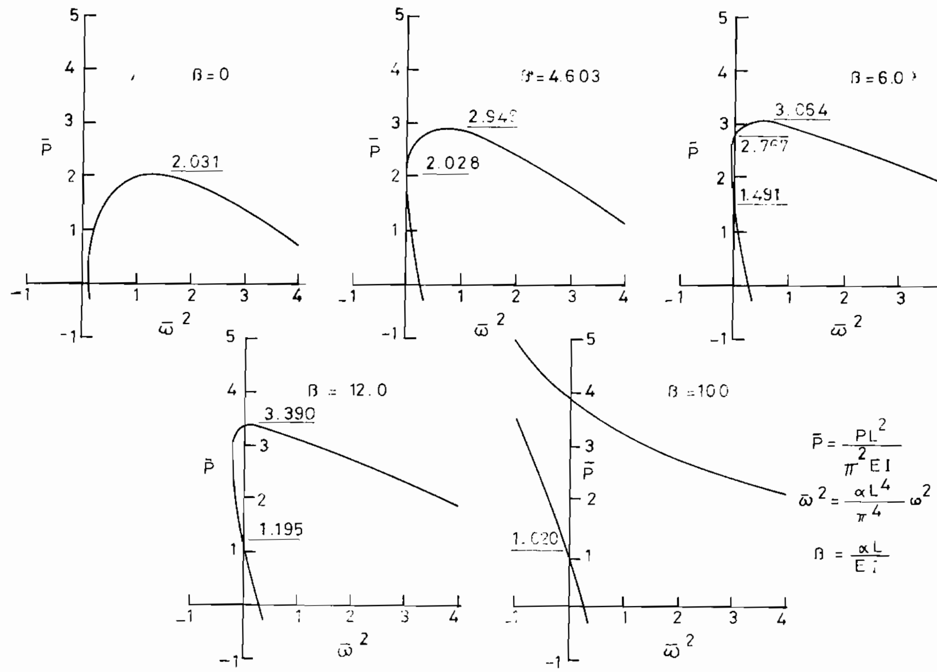


**Fig. 2.** Load versus frequency curves for particular values of $\beta$.

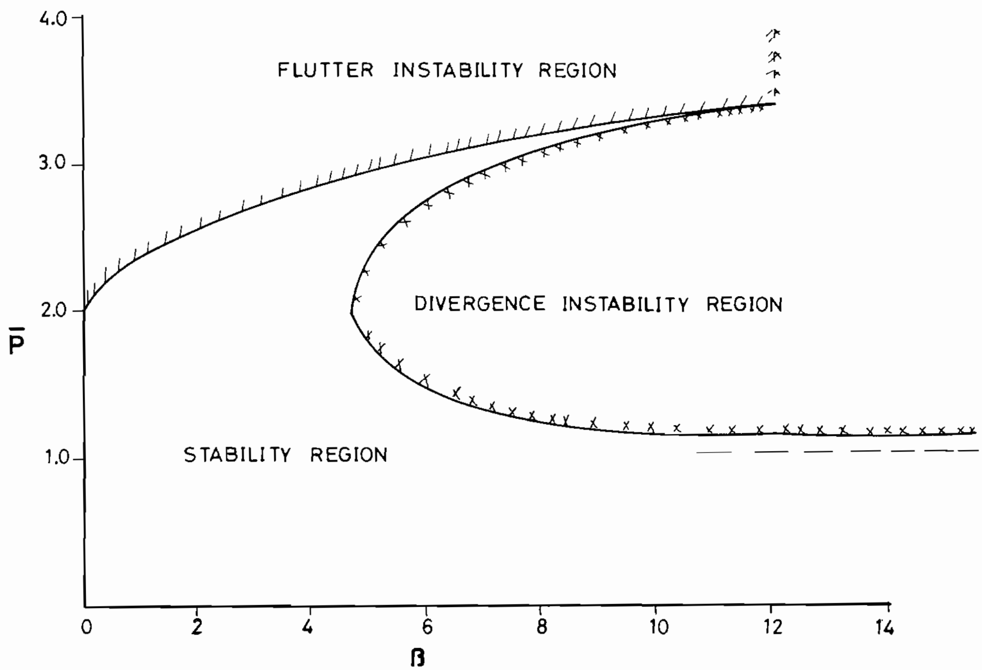


**Fig. 3.** Stability diagram.

$\bar{P} = 3{\cdot}390$, $\bar{\omega}_1^2 = \bar{\omega}_2^2 = 0$. For values of $\beta$ greater than 12·0, therefore, instability occurs through divergence only, and the frequency curves are similar to the hyperbola of the first type described above. The curves for $\beta = 100$ show that as $\beta$ increases $\bar{P}$ approaches unity.

As shown in Fig. 3, for $0 \leqslant \beta < 4{\cdot}603$ only flutter instability occurs. The effect of the rotational spring in this interval is to increase the critical load as the spring's stiffness increases.

For $4{\cdot}603 \leqslant \beta \leqslant 12{\cdot}0$, as $P$ is increased, first divergent instability occurs, then there is a region of divergent instability followed by a region of stability until flutter instability occurs. As noted in this interval, for an increase in $\beta$ the divergent instability critical load decreases while the flutter instability critical load increases.

For $\beta > 12{\cdot}0$ only divergent instability occurs.

## REFERENCES

**Bolotin, V.V. 1963.** Nonconservative problems of the theory of elastic stability (English translation edited by G. Hermann). Pergamon Press.

**Hermann, G. 1967.** Stability of equilibrium of elastic systems subjected to nonconservative forces. Applied Mechanics Review **20:** 2: 103–8.

**Hermann, G. & Bungay, R.W. 1964.** On the stability of elastic systems subjected to nonconservative forces. Trans. ASME **86:** Series E: 436–40.

**Huang, N.C., Nachbar, W. & Nemat-Nasser, S. 1967.** On Willem's experimental verification of the critical load in Beck's Problem. Trans. ASME **89:** Series E: 243–245.

**Ziegler, H. 1968.** Principles of structural stability. Blaisdell Publishing Company, Waltham, Massachussets, U.S.A.

(*Received* 28 *April* 1976)

# عدم استقرارية عمود مثبت بمرونة وتحت تأثير قوة تابعة

جاسم الصايغ
قسم الهندسة المدنية بجامعة الكويت

بول كريستيانو
قسم الهندسة المدنية بجامعة كارنيجى ميلون ،
بتسبرج ، بنسلفانيا

## خلاصـــــة

في هذا البحث دراسة نظرية لعمود على شكل رافعة وواقع عند حافته تحت تأثير لولب زنبركى دورانى متبوع بقـوة . ووجـد الباحثـان أن نوع عدم الاستقـرار الناتـج يتـوقف على صلابة اللولب الزنبركى .