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A note on the number of normal subgroups of a group

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ABSTRACT

In this paper it is shown that there is a one-to-one correspondence between the family of all normal subgroups of a group G and the family of all topologies that can be defined on G so that G is a locally connected and zero-dimensional topological group.

INTRODUCTION

Hewitt & Ross (1979) showed that if a group G is finite then the number of normal subgroups of G is the same as the number of topologies on G which makes G a topological group. In this paper we generalise this theorem to groups which are not necessarily finite.

Let G be a group and K a normal subgroup of G. Let $B(K) = \{xK | x \in G\}$. Then we have the following lemma.

Lemma 1. Let G be a group and K a normal subgroup of G. Then $B(K)$ is a base for a topology on G which makes G a locally connected and zero-dimensional topological group.

Proof. It is easy to check that $B(K)$ is a base for a topology $t(K)$ on G. Furthermore, G endowed with this topology is a topological group. Since the family of cosets $\{x\, K | x \in G\}$ decomposes G into disjoint open sets, it follows that xK is both open and closed for each $x \in G$. Hence G is zero-dimensional. To show that G is locally connected we will show that K is a connected subset of G. If V is an open subset of K, let $x \in V$, then $xK \subset V$. Hence $K = V$. This completes the proof of the lemma.

Theorem 1. Let G be a group. Then there is a one-to-one correspondence between the family of all normal subgroups of G and the family of all topologies which makes G a locally connected and zero-dimensional topological group.

Proof: Let $\mathscr{K}$ be the family of all normal subgroups of G and τ be the family of all topologies on G which makes G a locally connected and zero-dimensional topological group. For each $K \in \mathscr{K}$ let $t(K)$ be the topology on G generated by the base $B(K)$. Define a function $\psi : \mathscr{K} \to \tau$ by $\psi(K) = t(K)$. Clearly ψ is injective. To show that ψ is surjective, let $t \in \tau$. Let G_0 be the component of the identity. Then G_0 is a normal subgroup of G. We will show that $B(G_0)$ is a base for t. Since t is locally connected, xG_0 is open for each $x \in G$. Thus $B(G_0) \subset t$.

Now let $V \in t$ and let $x \in V$. Claim that $x\, G_0 \subset V$ (for otherwise the open set $U = V \cap x\, G_0 \subset x\, G_0$). Since G is zero-dimensional, U contains an open and closed subset. This contradicts the fact that G_0 is connected. Hence $\psi(G_0) = t$.

This completes the proof of the Theorem.

Consider now the case when the group G is finite.

Lemma 2. Let G be a finite topological group. Then G is locally connected and zero-dimensional.

Proof: Let G_0 be the component of the identity of G. Then the family $B(G_0)$ is a base for the topology on G. Now G_0 is the smallest open neighbourhood of the identity. For if V is an open neighbourhood of the identity such that $V \subset G_0$, then, of course, it is also a compact open neighbourhood of the identity. Then there exists a symmetric open neighbourhood U of the identity such that $V\,U \subset V$ (Hewitt & Ross 1979, Theorem 7.5). Then $U \subset e\,U \subset V\,U \subset V$ and $U^2 \subset V\,U \subset V$. By induction on n, we have $U^n \subset V$. Let

$$H = \bigcup_{n=1}^{\infty} U^n$$

Then H is an open subgroup of G_0 and hence is also closed. Hence $H = G_0$, and hence $V = G_0$. Consequently, G is locally connected.

Theorem 2. Let G be a finite group. Then the number of normal subgroups of G is the same as the number of all topologies on G which makes G a topological group.

Proof: Use Theorem 1 and Lemma 2.

Remark. Let G be a topological group with a topology t. It is still an open question whether there exists a subset K of G, with some algebraic and topological structures, that generates t in the sense that was described in Theorem 1. This subset K must at least satisfy the normality condition, i.e. $x^{-1}\,K\,x \subset K$ for all $x \in G$ to ensure the continuity of the inverse function $x \to x^{-1}$.

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REFERENCE

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نبين في هذا البحث انه يوجد تقابل بين عائلة الزمر الجزئية اللا متغيرة لزمرة G وعائلة كل التوبولوجيات التى يمكن تعريفها على G لتصبح زمرة توبولوجية صفرية البعد وموضعية الترابط .