J. Univ. Kuwait (Sci.) **4**, 1977

5-order local diffeomorphism

ADNAN AL-AQEEL

Department of the Mathematics, University of Kuwait

ABSTRACT

In this paper, the relation between almost complex manifolds, with almost complex structure J, and local diffeomorphisms of order 5 is found. The necessary and sufficient condition for these diffeomorphisms to be locally almost complex is also found.

INTRODUCTION

In Gray (1971), 3-order local diffeomorphisms associated with almost complex manifolds were introduced. This gave the way to define 3-symmetric spaces differently from that done in Ledger & Obata (1968). In this paper we will follow a similar technique done in Gray (1971) to find the relation between 5-order local diffeomorphisms and almost complex manifolds.

ALMOST COMPLEX MANIFOLDS AND LOCAL DIFFEOMORPHISMS

PROPOSITION 1

Let M be a C^∞ almost complex manifold with almost complex structure J. Assume that at each point $p \in M$, $M_p = \oplus M_{pi}$ $(i = 1, 2)$, such that $J_p(M_{pi}) = M_{pi}$. Then there exists a neighbourhood U_p of p and a diffeomorphism $s_p: U_p \to U_p$ such that

(i) p is the only isolated fixed point of s_p.

(ii) $s_p^5 = id$ in U_p.

Proof: Let I be the identity tensor over M. Put

$$S = \oplus S_i, \quad i = 1, 2,$$

where $S_i = \left(\cos \dfrac{2\Pi i}{5}\right) I + \left(\sin \dfrac{2\Pi i}{5}\right) J$

Then since $J_p(M_{pi}) = I_p(M_{pi}) = M_{pi}$, we have $S_{pi}(M_{pi}) = M_{pi}$.

Consequently $S^5 = I$.

Let

$$s_p = \exp_p \oplus S_p \oplus \exp_p^{-1}$$

where $\exp_p$ denotes the exponential map of M at p, $\exp_p$ is a diffeomorphism from a neighbourhood V near the zero vector in M_p to a neighbourhood of p in M, S_p does not have 0 or 1 as an eigenvalue.

Therefore, $s_p: U_p \to U_p$ is a diffeomorphism having p as the only isolated fixed point in U_p. We also have, maybe in a smaller neighbourhood,

$$s_p^5 = 1_{U_p} \qquad //$$

To prove the converse of proposition 1, we need the following definition.

Definition: Let M be a C^∞ manifold such that at each point $p \in M$, $M_p = \oplus M_{pi}$, $(i = 1, 2)$. A family of local 5-order diffeomorphisms on M is a C^∞ map $p| \to s_p$, which assigns to each $p \in M$ a local diffeomorphism $s_p : U_p \to U_p$, such that

(i) p is the only isolated fixed point of s_p.
(ii) $s_p^5 = 1_{U_p}$.
(iii) If ds_p is the induced map of s_p, we have $ds_p(M_{pi}) = M_{pi}$.

We will refer to the manifold in the above definition as a C^∞ manifold with a family of 5-order diffeomorphisms.

PROPOSITION 2

Let M be a C^∞ manifold with a family of local 5-order diffeomorphisms. Then there is a C^∞ almost complex structure J on M.

Proof: Let $p \in M$, $s_p: U_p \to U_p$ is a C^∞ diffeomorphism of 5-order in a neighbourhood U_p of p. Since p is an isolated fixed point of s_p, the induced map ds_p at p is a linear transformation of M_p, i.e.,

$$ds_p : M_p \to M_p, \quad \text{and} \quad (ds_p|_p)^5 = I_p$$

where I_p is the identity transformation on M_p.

Let $S_p = ds_p|_p$. We have $M_p = \oplus M_{pi}$, and $S_p(M_{pi}) = M_{pi}$ $(i = 1, 2)$
Therefore, $S_p = \oplus S_{pi}$ $(i = 1, 2)$, where $S_{pi} : M_{pi} \to M_{pi}$.
Write

$$S_{pi} = \left(\cos \frac{2\Pi i}{5}\right) I_{pi} + \left(\sin \frac{2\Pi i}{5}\right) J_{pi}$$

where $I_{pi} : M_{pi} \to M_{pi}$ is the identity.

Since S_p does not have 0 or 1 as an eigenvalue, we must have

$$J_{pi} : M_{pi} \to M_{pi}, \text{ such that } J_{pi}^2 = -I_{pi}$$

so that

$$(S_{pi})^5 = \left[\left(\cos \frac{2\Pi i}{5}\right) I_{pi} + \left(\sin \frac{2\Pi i}{5}\right) J_{pi}\right]^5 = I_{pi}$$

Now $S_p^5 = \oplus (S_{pi})^5 = I_p$. Let $J_p = \oplus J_{pi}$.

Therefore, $J_p: M_p \to M_p$ is an almost complex structure on M_p. This is true for all points in M. Hence, we have an almost complex structure J on M. J is C^∞ because $p \mapsto S_p$ is C^∞. //

Remark 1

In propositions 1 and 2 we assumed that we have two differentiable distributions M_1 and M_2 over M, such that for each $p \in M$, $M_p = M_{p1} \oplus M_{p2}$. The reason for this is that when we later define Riemannian 5-symmetric spaces, we want the symmetry tensor field S of type (1,1) on M, where $S_p = ds_p|_p$ to have two pairs of complex eigenvalues, and this gives rise to two differentiable distributions M_1 and M_2 over M (see Gray 1971). If S has one pair of complex eigenvalues w and $\bar{w}$ such that $w^2 \neq \bar{w}$, then it is proved in Ledger & Obata (1968) that M admits the structure of a locally symmetric space, and we are not interested in that.

Remark 2

A C^∞ manifold with a family of 5-order diffeomorphisms give rise that at each point $p \in M$ we have a local diffeomorphism s_p in a neighbourhood of p, and from proposition 2, we have an almost complex structure J over M, but it is false that s_p is almost complex in U_p, i.e., it is false that in U_p we have $ds_p \circ J = J \circ ds_p$. The following proposition will give the necessary and sufficient condition for that.

PROPOSITION 3

Let M be analytic manifold with a family of 5-order difleomorphisms, each of which is an isometry. Suppose that M is almost Hermitian with respect to the almost complex structure J of the family, then each s_p is almost complex isometry if and only if S preserves $\nabla^k J$ for $k = 0, 1, 2, \ldots$, where S is the tensor field of type (1,1) such that $S_p = ds_p|_p$.

Proof: (i) If each s_p is an almost complex isometry, then it is obvious that S, where $S_p = ds_p|_p$, preserves $\nabla^k J$ for $k = 0, 1, 2. \ldots$

(ii) Suppose that S preserves $\nabla^k J$ for $k = 0, 1, 2, \ldots$ then for all $p \in M$, we have $ds_p[(\nabla^k J)_p] = (\nabla^k J)_p$, $k = 0, 1, 2, \ldots$.

Let $\tilde{J} = s_p(J)$, it is well known that $\tilde{J}$ is a new almost complex structure over M, and since s_p is an isometry, M is almost Hermitian with respect to $\tilde{J}$. $\tilde{J}$ is analytic, since J and s_p are analytic, therefore J and $\tilde{J}$ can be expanded in power series at p in $(\nabla^k J)_p$ and $(\nabla^k J)_p$ respectively, but $(\nabla^k J)_p = (\nabla^k \tilde{J})_p$ since s_p is an isometry and $S_p = ds_p|_p$ preserves $\nabla^k J$. Hence $J = \tilde{J}$ in a neighbourhood of p, and so s_p is an almost complex isometry. //

This proof is patterned after that of Gray (1971). We are now in the position to define Pseudo-Riemannian 5-symmetric space.

DEFINITION

A Pseudo-Riemannian locally 5-symmetric space M is a C^∞ manifold with a family of 5-order diffeomorphisms $p \mapsto s_p$ each of which is an almost complex structure of the family.

REFERENCES

Gray, A. 1971. Riemannian manifolds with geodesic symmetries of order 3. J. Diff. Geom. **5**: 343–69.
Ledger, A.J. & Obata, M. 1968. Affine and Riemmannian S-manifolds. J. Diff. Geom. **2**: 451–9.

التخالف المحلى ذو الرتبة الخامسة

عدنان العقيل
قسم الرياضيات بجامعة الكويت

خلاصة

لقد امكن التوصل ، في هذا البحث ، الى تعيين العلاقة ما بين البنية قريبة التعقد والمتنوعات قريبة التعقد والتخالفات المحلية من الرتبة الخامسة . كما امكن ايجاد الشرط الضرورى والكافي لهذه التخالفات ان تكون قريبة التعقد محليا .