THE JOURNAL OF THE UNIVERSITY OF KUWAIT (Science)

VOL. 4, 1977

Contents

MATHEMATICS

BOTANY

ZOOLOGY

GEOLOGY

J. Univ. Kuwait (Sci.) **4**, 1977

Umbilical hypersurfaces of riemannian, Kähler and nearly Kähler manifolds

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ABSTRACT

The results obtained by Tachibana and Kashiwada concerning the characterization of spaces of constant holomorphic sectional curvature in terms of geodesic hyperspheres are extended in this paper.

1. INTRODUCTION

This work originated with an examination of results previously obtained by Tachibana & Kashiwada (1973) which were concerned with characterisation of spaces of constant holomorphic sectional curvature in terms of geodesic hyperspheres. In particular they proved:

Theorem 1. In an n ($\geqslant 3$)*-dimensional connected riemannian Einstein space* M^n, *if each geodesic hypersphere at any point is totally umbilic, then* M^n *is a space of constant curvature.*

In addition they proved the following analogous theorem for Einstein–Kähler manifolds, which we state in terms of their somewhat complicated notation.

Theorem 2. Let M^{2n} ($n \geqslant 2$) *be a connected Einstein–Kähler manifold with metric* g_{ij} *and complex structure* $F_i{}^h$ *which is parallel and satisfies*

$$F_i^r F_r^h = -\delta_i^h, \quad F_i^r F_j^s g_{rs} = g_{ij}, \quad i, j, r, s = 1, \ldots, 2n.$$

Let S *be a geodesic sphere centred at some arbitrary point 0. Let* $(B^i{}_a)$, $a = 1, 2, \ldots, 2n-1$, *be the Jacobian matrix of the coordinate transformation expressing* S *as a submanifold of* M^{2n}. *Let* g_{ab} *be the induced metric on* S, *and let* N^i *be the local unit vector field normal to* S. *Let* η_a *be defined by*

$$\eta_a = B_a^i F_i^h N_h.$$

Let H_{ab} *be the coefficients of the second fundamental form of* S. *Then if* M^{2n} *has constant holomorphic sectional curvature, each such geodesic sphere is such that*

$$H_{ab} = \mu g_{ab} + \nu \eta_a \eta_b$$

where μ, ν *are functions such that* $\mu\nu$ = *constant. Conversely, if this property holds for* *every geodesic sphere, the manifold* M^{2n} *must have constant holomorphic sectional curvature.*

In this paper we obtain sharper theorems—in particular we show that the Einstein requirement is unnecessary in Theorems 1 and 2, that the rather unnatural condition $\mu\nu$ = constant of Theorem 2 is unnecessary and the results apply to nearly Kähler as well as to Kähler manifolds.

2. THE RIEMANNIAN CASE

By an *r-dimensional linear element* E_r of a differential manifold M where dim $M = n$ we mean a point P of M and an r-dimensional subspace of the n-dimensional tangent space to M at P.

We recall the following theorem essentially due to Schouten (1921)

Theorem 3. Let M *be a riemannian manifold of dimension* $n>3$. *Then a necessary and sufficient condition that to any* $(n-1)$*-dimensional linear element* E_{n-1} *of* M *there is an umbilical hypersurface tangent to* E_{n-1}, *is that* M *is locally conformally flat.*

This proof depends upon the following (see, for example Schouten & Struik (1932)):

Lemma 1. An umbilical submanifold of a riemannian manifold M *remains umbilical under a conformal change of metric of* M.

We observe that the following theorem is a sharpening of Theorem 1 since the Einstein condition now follows as a consequence of the theorem and is no longer part of the hypothesis.

Theorem 4. Let M *be a connected* C^∞*-riemannian manifold of dtmension* $\geqslant 3$. *Then every sufficiently small geodesic hypersurface is totally umbilical if and only if* M *is of constant curvature.*

We originally proved this theorem by the method of moving frames, making use of Theorem 3 and Lemma 1. We were informed by Kulkarni that he independently had recently proved this theorem by an analogous method so we omit the proof and refer the reader to Kulkarni (1975).

Theorem 4 is no longer true when dim $M = 2$ because any curve on a 2-dimensional manifold is trivially totally umbilic. However, the following theorem is valid when dim $M \geqslant 2$:

Theorem 5. Let M *be a connected riemannian* C^∞*-manifold with dim* $M \geqslant 2$. *Then every sufficiently small geodesic hypersphere has constant mean curvature if and only if* M *is a harmonic space. In particular if dim* $M = 2$ *or* 3, M *is of constant curvature.*

To prove the theorem, choose, for example, a set of normal coordinates centred at some point P_0 of M, and denote by $S(P_0,s)$ the geodesic sphere centred at P_0 and of radius s. The second fundamental form relative to this system of coordinates is easily computed to be $s_{,ij}$ where comma denotes covariant differentiation with respect to the riemannian connection (see, for example, Dombrowski (1968)). The mean curvature is just $g^{ij}s_{,ij} = \Delta_2 s$. This is constant on the sphere $S(P_0,s)$ for all s if and only if

$$\Delta_2 s = \chi(P_0, s)$$

for some function χ. This, however, is precisely the condition that M is harmonic at

P_0 (Ruse, Walker & Willmore (1962), p. 35). Since P_0 is arbitrary, it follows that the space is harmonic at all points. Moreover it is known (p. 67 of the same book) that when dim $M = 2$ or 3, harmonic spaces are of constant curvature and conversely, and the theorem is therefore proved.

We note that when dim $M \geqslant 4$ there are harmonic spaces which are not of constant curvature, e.g. the complex projective spaces. Thus for $n \geqslant 4$ the requirement of constant mean curvature of geodesic spheres is less restrictive than the requirement of being totally umbilic. On the other hand when $n = 2$ it is more restrictive.

3. THE KÄHLER AND NEARLY KÄHLER CASES

We now pass to the consideration of Kähler and nearly Kähler manifolds. Let M be a riemannian manifold of real dimension $2n$, with a riemannian metric $(g_{ij}) = \langle\ \rangle$, and an almost complex structure J satisfying $J^2 = -$identity. We assume that the metric is almost hermitian, i.e. $\langle JX, JY\rangle = \langle X, Y\rangle$ for any two vector fields X, Y on M. We shall assume that the Kähler condition is satisfied, i.e. $\nabla_X J = 0$, where ∇ is the riemannian connection. We first wish to prove the Kähler equivalent of part of Theorem 3, which is as follows:

Theorem 6. Let M *be a Kähler manifold of complex dimension* $n \geqslant 4$. *Let us assume that corresponding to any real* $(2n-1)$*-dimensional linear element* E_{2n-1} *of* M *there is a hypersurface tangent to* E_{2n-1}, *which is quasi-umbilical with respect to the special direction* Je_n *where* e_n *is the locally defined unit normal vector field. Then* M *has zero Bochner curvature tensor.*

We first choose any orthonormal basis such that e_n is normal to the E_{2n-1} at P, and $e_1, e_{1*}, e_2, e_{2*}, \ldots, e_{n-1}, e_{n-1*}, e_{n*}$ are tangent. Here we have used the notation $e_{i*} = Je_i$, $i = 1, 2, \ldots, n$. Then we have for the metric of M,

$$ds^2 = \sum_i \omega_i^2,$$

$$d\omega_i = \omega_j \wedge \omega_{ij}, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$\omega_{ij} = \omega_{i^*j^*}, \quad \omega_{i^*j} = \omega_{j^*i},$$

the latter two equations reflecting the Kähler and the almost hermitian condition.

The direction of the E_{2n-1} at P is given by $\omega_n = 0$; we find

$$\omega_{\alpha n} = h_{\alpha\beta}\omega_\beta, \quad \alpha, \beta = 1, 1^*, \ldots, (n-1)^*, n^* \tag{3.1}$$

where $(h_{\alpha\beta})$ is the second fundamental form tensor.

We now impose the condition that the manifold M is quasi-umbilical (Chen 1973) with respect to the special tangent direction e_{n*}. This gives

$$\begin{aligned} h_{ab} &= A\delta_{ab}, \quad a, b = 1, 1^*, \ldots, (n-1)^*, \\ h_{an^*} &= 0, \\ h_{n^*n^*} &= B, \end{aligned} \tag{3.2}$$

for some functions A and B.

Exterior differentiation of (3.1) and use of the Cartan structural equations and (3.2) leads to

$$\Omega_{an} = dA \wedge \omega_a + (A-B)A\omega_{n*} \wedge \omega_{a*} + A^2 \omega_a \wedge \omega_n,$$
$$\Omega_{n*n} = dB \wedge \omega_{n*} + (A-B)A \sum_a \omega_a \wedge \omega_{a*} + B^2 \omega_{n*} \wedge \omega_n.$$

Substituting for the curvature tensor in the above equations leads to the following relations:

$$R_{anbc} = -A_b \delta_{ac} + A_c \delta_{ab},$$
$$-\tfrac{1}{2} R_{ann^*a} = A_{n*}, \quad -\tfrac{1}{2} R_{ann^*a^*} = (A-B)A,$$
$$\tfrac{1}{2} R_{ann^*c} = 0 \quad \text{for} \quad c \neq a^*, a,$$
$$R_{n^*naa^*} = 0,$$
$$R_{n^*nbc} = 0,$$
$$R_{n^*nn^*c} = B_c.$$

These relations could also have been obtained directly from the classical Codazzi equation. Let $n \geqslant 4$ and take a, b, c in such a way that e_n, e_a, e_b, e_c span an anti-holomorphic 4-space. It follows that

$$R_{anbc} = 0.$$

It follows from Yano & Sawaki (1975) that M is Bochner flat, and the theorem is proved.

We know that a Bochner flat Einstein space is of constant holomorphic sectional curvature. It follows that Theorem 6 implies Theorem 2 of Tachibana & Kashiwada (1973) in the case $n \geqslant 4$ with their Einstein condition but without their further assumption $\mu\nu$ = constant (corresponding to AB = constant in our notation).

However, in the case $n = 2$ or $n = 3$, although we cannot use the previous argument involving an anti-holomorphic 4-space, we can still obtain the result of constant holomorphic sectional curvature provided we assume that the space is Einstein and, in addition that AB = constant. This follows because we have

$$A_{n*} = 0, \quad (2n-3)A_b + B_b = 0.$$

The assumption AB = constant now gives A = constant, B = constant, and hence

$$R_{abnc} = 0 \quad \text{for} \quad b \neq a^*, \quad R_{abnn^*} = 0.$$

The conclusion of constant holomorphic sectional curvature now follows as before.

However, we now show that in fact *both the Einstein condition and the condition* AB = *constant can be relaxed from Theorem 2.* As in the analogous proof of Theorem 4 (see Kulkarni 1975) we take a fixed point P and a unit tangent vector e_n at P. We take a point Q on the uniquely determined geodesic at P in this initial direction, and we consider a set of moving frames so that e_n is along geodesics emanating from Q. Using analysis analogous to the proof of Theorem 6, the new relation $\omega_n = ds$ leads to the following additional relations satisfied by the curvature tensor:

$$R_{anan} = A_n - A^2; \quad R_{ancn} = 0 \text{ if } a \neq c; \quad R_{ann^*n} = 0;$$
$$R_{n^*ncn} = 0; \quad R_{n^*nn^*n} = (B_n - B^2).$$

We now appeal to a result by Tanno (1973).

Theorem 7. Let dim M = m = *2n ⩾ 4. Assume that the almost Hermitian manifold* (M,g,J) *satisfies*

$$(*) \qquad g(R(JX,JY)JX,JZ) = g(R(X,Y)X,Z)$$

for all tangent vectors X, Y *and* Z. *Then* (M, g,J) *is of constant holomorphic sectional curvature at* x *if and only if* R(X,JX)X *is proportional to* JX *for every tangent vector* X *at* x.

Condition (*) is satisfied in every Kähler manifold, so it is sufficient in our case to prove that the second condition is satisfied. If we take for X the arbitrary vector e_n, then

$$R(e_n, e_{n^*})e_n = R_{ABnn^*}\,\delta_n^B\, e_A$$
$$= R_{Annn^*}\, e_A = R_{n^*nnn^*}\, e_{n^*}.$$

It follows that the space is of constant holomorphic sectional curvature at P, and since the manifold is Kähler the result follows from the corresponding Schur theorem. Thus we have proved

Theorem 8. Let M *be a connected Kähler manifold such that any geodesic sphere is quasi-umbilical with respect to the tangent vector* Je_n, *where* e_n *is the unit normal. Then if dim* M$\geqslant$*4, the manifold is of constant holomorphic sectional curvature.*

We now obtain an extension of Theorem 8. Suppose that M is any almost Hermitian manifold satisfying the condition (*), with real dim $M = 2n \geqslant 4$, but at this stage we do not make use of the Kähler condition.

Then we obtain in the same manner

$$\Omega_{n^*n} = (B_n - B^2)\omega_n \wedge \omega_{n^*} + B_a\omega_a \wedge \omega_{n^*} + (B-A)\omega_a \wedge \omega_{an^*}.$$

Put now

$$\omega_{an^*} = \gamma_{ab}\omega_b + \gamma_{an}\omega_n + \gamma_{an^*}\omega_{n^*}.$$

Then we get

$$R_{n^*nnn^*} = B^2 - B_n,$$
$$R_{n^*nan} = (A-B)\gamma_{an},$$
$$R_{n^*nan^*} = (A-B)\gamma_{an^*} - B_a.$$

The condition (*) implies at once $B_a = 0$. Hence

$$R(e_n, e_{n^*})e_n = -(A-B)\gamma_{an}e_a + (B^2 - B_n)e_{n^*}.$$

Now we have

$$\omega_{an^*} = \langle \nabla e_a, e_{n^*} \rangle$$
$$= -\langle \nabla e_{a^*}, e_n \rangle + \langle (\nabla J)e_a, e_n \rangle$$
$$= \omega_{na^*} + \langle (\nabla J)e_a, e_n \rangle$$

and so

$$\omega_{an^*} = -A\omega_{a^*} + \langle (\nabla J)e_a, e_n \rangle.$$

Thus

$$\gamma_{an} = \omega_{an^*}(e_n) = \langle (\nabla_{e_n} J)e_a, e_n \rangle.$$

Suppose now that M *is a nearly Kähler manifold,* i.e. $(\nabla_X J)X = 0$ (Gray 1970). For such a manifold (*) is satisfied. Further

$$\langle(\nabla_{e_n}J)e_a, e_n\rangle = -\langle(\nabla_{e_a}J)e_n, e_n\rangle.$$

Since $\langle e_{n*}, e_n\rangle = 0$ we obtain

$$0 = \nabla_{e_a}\langle Je_n, e_n\rangle = \langle(\nabla_{e_a}J)e_n, e_n\rangle - \langle\nabla_{e_a}e_n, e_{n*}\rangle + \langle e_{n*}, \nabla_{e_a}e_n\rangle$$

or

$$\langle(\nabla_{e_n}J)e_a, e_n\rangle = 0.$$

This implies $\gamma_{an} = 0$ and $R(e_n, e_{n*})e_n = (B^2 - B_n)e_{n*}$. It follows from Tanno's theorem that M has constant holomorphic sectional curvature. Using a classification theorem of Gray (1974) we obtain what we claim is an appropriate replacement of Theorem 2, namely:

Theorem 9. Let M *be a nearly Kähler manifold such that any geodesic sphere is quasi-umbilical with respect to the tangent vector* Je_n*, where* e_n *is unit normal. Then if dim* $M \geqslant 4$*, the manifold has constant holomorphic sectional curvature. Moreover* M *is locally isometric to a complex space form or to* S^6.

In conclusion we would thank the referee for suggesting several improvements in the presentation of this paper.

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(*Received* 20 *March* 1977)

متنوعات فوق السطوح السرية لريمان وكوهلر والقريبة منها

ل . ڤان هيكه
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الجامعة الكاثوليكية في
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ت . ج . ويلمور
قسم الرياضيات بجامعة
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خلاصة

نتيجة للدراسات التي اجريت بشأن السطوح ذات التقويس المقطعي المتكامل بدلالة فوق الكرات الجيوديسية ، امكن تعيين خواص الفضاء ذى التقويس الثابت . كما امكن الحصول على نظريات اقوى بشأن متنوعات كوهلر والمتنوعات القريبة منها .