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On disconjugacy of linear differential equations of the second order

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ABSTRACT

The object of this paper is to give criteria for the disconjugacy of the equation

$$[p(t)x']' + q(t)x = 0.$$

Some other related results are obtained.

1. INTRODUCTION

Consider the equation

$$[p(t)y']' + q(t)y = 0, \quad (') \equiv \frac{d}{dt} \tag{1}$$

where $p(t)$ is a continuous differentiable positive function on an interval I and $q(t)$ is continuous on I. Equation (1) is disconjugate on I if every nontrivial solution has at most one zero on I (Coppel 1971, Wintner 1951).

The problem of disconjugacy for eqn (1) has been studied by Hartman & Wintner (1949), Leighton (1952, 1962), Potter (1953), Moore (1955), Barrett (1959), Reid (1971) and others. Many other references can be found in the reviews by Swanson (1968), Willet (1969), Barrett (1969) and Hartman (1973).

Several types of criteria were found for disconjugacy of (1). Some of these criteria depend on the associated functional

$$Q[\eta] = \int_a^b (p\eta'^2 - q\eta^2)dt,$$

(see Reid 1971). Criteria of integral type were obtained by Hille (1948), Borg (1949), Taam (1952) and Wong (1969). The first comparison criterion is due to Sturm (1836). More comparison criteria can be found in Swanson (1968).

It is the object of this paper to give new criteria to detect the disconjugacy of (1), where the other known criteria are not applicable.

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2. DISCONJUGACY CRITERIA

Consider the equation

$$[k(t)x']' + l(t)x = 0, \tag{2}$$

where $k(t)$ and $l(t)$ are continuous functions on $[a,b]$ with $k(t) > 0$. Let $x(t)$ be a solution of (2) with two consecutive zeros a and b, $a < b$. One can choose $x(t)$ to be positive on (a,b) and has the form

$$x(t) = h(t)g(t), \tag{3}$$

where $h(a) = h(b) = 0$, $h(t) > 0$ on (a,b) and $g(t) > 0$ on $[a,b]$. Since $g(t)$ solves the differential equation

$$(kh^2x')' + k[lh + (kh')']x = 0 \tag{4}$$

we conclude that (4) and (2) are disconjugate.

Comparing (4) with (1) and applying standard comparison theorems to ensure that also (1) is disconjugate, we obtain some criteria for the disconjugacy of (1).

Thus applying Sturm's comparison criterion (Sturm 1836) on (4) and (1) we obtain the following disconjugacy criterion.

Theorem 2.1. Equation (1) is disconjugate on $I = (a,b)$ if

$$p \geqslant kh^2 > 0 \tag{5}$$

and

$$q \leqslant h[lh + (kh')'] \tag{6}$$

for all t in I, with $h(a) = h(b) = 0$ when $b < \infty$ and $h(a) = 0$ when $b = \infty$.

Remark 2.1. Varying the choice of $h(t)$ we get various forms of (5) and (6). For example one can put $h(t) = (t-a)(b-t)$ or $h(t) = \sin \pi(t-a)/b-a$.

As an application of the previous criterion we give the following examples.

Example 1. The equation

$$x'' + \pi^2 x = 0$$

possesses a solution $x = \sin \pi t$, with two consecutive zeros $a = 0$ and $b = 1$. We use this equation and Theorem 2.1 to discuss the disconjugacy property for the equation

$$[(t+\tfrac{1}{3})(\tfrac{4}{3}-t)y']' + t(\lambda - t)(\mu + \nu \cos 2\pi t)y = 0. \tag{7}$$

Let $h(t) = t(1-t)$, $k(t) = 1$, $l(t) = \pi^2$ and $(a,b) = (0,1)$, conditions (5) and (6) take the forms

$$(t+\tfrac{1}{3})(\tfrac{4}{3}-t) \geqslant t^2(1-t)^2 > 0$$

and

$$t(\lambda - t)(\mu + \nu \cos 2\pi t) \leqslant t(1-t)[\pi^2 t(1-\mathrm{t}) - 2].$$

It is not difficult to conclude from these inequalities that (7) is disconjugate on (0,1) provided that

$$\lambda < 1, \quad \mu+\nu < -2, \quad \mu-\nu < \frac{\pi^2}{4}-2.$$

Remark. We notice that Sturm's comparison theorem cannot be applied.

Example 2. Consider the equation

$$(e^{\delta t}y')' + \frac{\alpha t^2 \sin \beta t}{(t+1)^\gamma} y = 0, \quad t \geq 0, \quad \delta > 0, \tag{8}$$

where α, β, γ and δ are constants. Applying Theorem 2.1 on (8) and the equation

$$x'' + \frac{1}{4(t+1)^2} x = 0,$$

with $a=0$, $b=\infty$ and $h(t)=t$, conditions (5) and (6) become

$$e^{\delta t} \geqslant t^2 > 0$$

and

$$\frac{\alpha t^2 \sin \beta t}{(t+1)^\gamma} \leq \frac{t^2}{4(t+1)^2}.$$

Solving these inequalities we get that (8) is disconjugate on $(0,\infty)$ for all β provided that

$$\gamma > 2, \quad |\alpha| < \tfrac{1}{4}, \quad \text{and } \delta > \frac{2}{e}.$$

Now we can obtain another criterion applying Taam's comparison theorem (Coppel 1971, p. 16) on (4) and (1).

Theorem 2.2. Equation (1) is disconjugate on $I=(a,b)$ if (2) is disconjugate and

(i) $$p \geq kh^2 > 0$$

(ii) q and $h[lh+(kh')']$ are integrable on I.

(iii) $$\int^b \frac{\mathrm{d}t}{kh^2} = \infty$$

(iv) $$\left| \int_t^b q\,ds \right| < \int_t^b h[lh+(kh')']dt$$

for all t in I.

Choosing $h(t)=(t-a)(b-t)$ and $k(t)\equiv 1$. One concludes the following conditions for the disconjugacy of (1):

(i) $$p > \left(\frac{b-a}{2}\right)^4,$$

(ii) q and lh^2 are integrable on I,

(iii) $$\left| \int_t^b q(s)ds \right| < \int_t^b lh^2 ds - \frac{(b-a)^3}{2}.$$

At the end of this section we give a useful property of the function $g(t)$.

Theorem 2.3. If

$$f(a)\cdot f(b) > 0,$$

where $f(t)=[k(t)h'(t)]'$, then $g'(d)=0$ for some $d\in(a,b)$.

Proof. From (3) we have

$$0 < x'(a) = h'(a)g(a) \Rightarrow h'(a) > 0$$

$$0 > x'(b) = h'(b)g(b) \Rightarrow h'(b) < 0$$

and

$$\left.\begin{aligned} f(a)g(a)+2h'(a)g'(a)k(a) &= 0 \\ f(b)g(b)+2h'(b)g'(b)k(b) &= 0 \end{aligned}\right\}(*)$$

Now since $f(a)f(b)>0$, then $f(a)$ and $f(b)$ are both negative or both positive. In the first case we conclude that the first summands of (*) are negative, therefore the second terms must be positive. Since $h'(a)>0$, $h'(b)<0$, $k(a)>0$ and $k(b)>0$, it follows that $g'(a)>0$ and $g'(b)<0$. Therefore $g'(t)$ vanishes at some point $d\in(a,b)$.

The case when $f(a)$ and $f(b)$ are both positive can be treated in a similar way and the proof is thus complete.

3. QUADRATIC FUNCTIONAL

A function $\eta(t)$ defined on $[a,b]$ is called admissible if $\eta(a)=\eta(b)=0$ and $\eta(t)\in C^1[a,b]$. Consider the quadratic functional

$$Q[\eta] = \int_a^b \{p(t)\,(\eta'(t))^2 - q(t)\,(\eta(t))^2\}dt. \qquad (10)$$

The function $Q[\eta]$ is called positive if $Q[\eta]>0$ for all nontrivial admissible functions $\eta(t)$.

Consider the equation

$$y''+q(t)y = 0, \qquad (11)$$

where $q(t)\in C[a,b]$. Applying the transformation

$$\eta(t) = r(t)\xi(t), \qquad (12)$$

where $r(t)$ and $\xi(t)\in C^2[a,b]$ to get

$$Q[\eta] = \int_a^b (r'^2\xi^2+2rr'\xi\xi'+r^2\xi'^2-qr^2\xi^2)dt.$$

Integration by part implies

$$Q[\eta] = \int_a^b [r^2\xi'^2-r\xi^2(r''+qr)]dt. \qquad (13)$$

From (13) and Theorem 7 of Coppel (1971, p. 10) we get the following theorem.

Theorem 3.1. If there exists a function $r(t)\in C^2[a,b]$ such that

$$r(r''+qr) < 0, \tag{14}$$

for all $t\in[a,b]$, then (11) is disconjugate in $[a,b]$.

Example 3. Consider the equation of the deflection curve of a buckled bar with continuously varying cross section (Timoshenko & Gere 1961, p. 125)

$$y''+\frac{k^2}{t^2}y = 0, \quad t > 1.$$

Let $r(t)=t^{\alpha}$, $(0<\alpha<1)$. Condition (14) takes the form

$$t^{2(\alpha-1)}\,[k^2-\alpha(1-\alpha)] \leqslant 0.$$

This inequality holds if

$$k^2 \leqslant \alpha(1-\alpha).$$

Since the expression $\alpha(1-\alpha)$ has a maximum of $\frac{1}{4}$ at $\alpha=\frac{1}{2}$, it follows that the considered equation is disconjugate on $[1,\infty)$ whenever $k^2\leqslant\frac{1}{4}$.

Theorem 3.2. Let $r(t)$ be a solution of

$$[p(t)r']'+m(t)r = 0, \tag{15}$$

where $m(t)\in C[a,b]$. A solution $y(t)$ of (11), such that $y(a)=0$, will vanish at some point d in (a,b), if

$$\int_a^b [(q-m)\xi^2-p\xi'^2r^2dt > 0 \tag{16}$$

for all $\xi(t)\in C^1[a,b]$ with

$$r(a)\,\xi(a) = r(b)\,\xi(b)=0,$$

provided that $r(t)\,\xi(t)$ is not a constant multiple of $y(t)$.

Proof. Suppose that $y(t)$ is a solution of (11), with $y(a)=0$. From proposition 3 of Coppel (1971, p. 9) we conclude that if there exists an admissible function $\eta(t)=r(t)\,\xi(t)$ such that

$$-Q[\eta] = \int_a^b \{qr^2\xi^2-p[(r\xi)']^2\} > 0, \tag{17}$$

then $y(d)=0$ for some point $d\in(a,b)$. Integrating the second term of (17) by parts we obtain

$$\int_a^b [qr^2\xi^2+\xi^2r(r'p)'-p\xi'^2\,r^2]dt > 0.$$

Since $r(t)$ satisfies (15) we get (16) from the last inequality.

Example 4. Consider the equation of the deflection curve of a buckled bar under distributed axial loads (Timoshenko & Gere 1961, p. 101)

$$y''+(t+1-k)y = 0, \quad 0 < k < \frac{\pi}{2}$$

and let $y(t)$ be a solution of this equation such that $y(0)=0$.

Taking $r(t) \equiv \sin t$, $p(t) \equiv 1$, $m(t) \equiv 1$, $\xi(t) \equiv 1$, $a=0$ and $b=\pi$ in Theorem 3.2, condition (16) reduces to the form

$$\int_0^\pi (t-k)\sin^2 t\, dt = \pi\left(\frac{\pi}{2}-k\right) > 0.$$

Consequently $y(d)=0$, where $0<d<\pi$.

It is not difficult to realise that neither the Sturm's theorem nor the Sturm–Picone theorem can be applied for the considered equation.

4. GENERATION OF DISCONJUGATE DIFFERENTIAL EQUATIONS

Let

$$[p_i(t)x']' + q_i(t)x = 0, \quad i = 1, 2, \ldots, n. \tag{18}$$

be a system of disconjugate differential equations on an interval $I=[a,\infty)$, where $p_i \in C^1[a,\infty)$, $p_i>0$ and $q_i \in C[a,\infty)$, $i=1, 2, \ldots, n$. One can ask the question: what is the class of functions $\phi_i(t)$, $i=1, 2, \ldots, n$, for which

$$\left\{\left(\sum_{i=1}^n \phi_i(t)\, p_i(t)\right] y'\right\}' + \left(\sum_{i=1}^n \phi_i(t)\, q_i(t)\right) y = 0 \tag{19}$$

is disconjugate on I?

This question has been studied for special cases by Adamov (1948), Petroparlovskaya (1955), Markus & Moore (1956), Kondratév (1957) and Willet (1969).

In the following we give a partial answer to the previous question.

Theorem 4.1. Let the system of eqns (18) be disconjugate on an interval $I=[a,\infty)$. Assume

$$\phi_i(t) > 0, \quad t \in I, \quad i = 1, 2, \ldots, n,$$

and

$$\sum_{i=1}^n \phi'_i(t)\, u_i \leq 0, \tag{20}$$

where u_i are continuous differentiable functions on (a,∞) with

$$u'_i + q_i + \frac{u_i^2}{p_i} \leq 0, \quad i = 1, 2, \ldots, n,$$

then (19) is disconjugate.

Proof. Applying Bôcher's theorem (Bôcher 1900–1) we conclude that there exist functions $u_i(t) \in C^1[a,\infty)$, $i=1, 2, \ldots, n$, such that

$$u'_i + q_i + \frac{u_i^2}{p_i} \leq 0, \quad a < t < \infty, \quad i = 1, 2, \ldots, n.$$

If we multiply these inequalities by δ_i and take the sum from $i=1$ to $i=n$ we get

$$\sum_{i=1}^n \phi_i u'_i + \sum_{i=1}^n \phi_i q_i + \sum_{i=1}^n \frac{\phi_i u_i^2}{p_i} \leq 0. \tag{21}$$

Cauchy's inequality together with the assumptions of the theorem imply

$$\left(\sum \phi_i u_i\right)' + \sum q_i \phi_i + \frac{(\sum \phi_i u_i)^2}{\sum \phi_i p_i} < 0. \tag{22}$$

Hence by Bôcher's theorem the disconjugacy of eqn (19) follows.

Theorem 4.2. If eqns (18) are disconjugate on $I=[a,b]$, then eqn (19) is disconjugate on I if

$$\phi_i(t) > 0, \quad i = 1, 2, \ldots, n \tag{23}$$

$$\left(\frac{p_i \phi_i'}{\phi_i}\right)' \geq 0, \quad i = 1, 2, \ldots, n \tag{24}$$

for all values of $t \in I$.

Proof. First we prove that the equation

$$(\phi_1 p_1 y')' + \phi_1 q_1 y = 0 \tag{25}$$

is disconjugate on I when $\phi(t)$ satisfies (23) and (24). Suppose that on the contrary there exists a solution $z(t)$ of (25) which vanishes at $t=\tau_1$ and $t=\tau_2$ on the interval I. It is clear that

$$\int_{\tau_1}^{\tau_2} (p_1 zz')' \, dt = 0.$$

On the other hand we have

$$0 = \int_{\tau_1}^{\tau_2} (p_1 zz')' \, dt = \int_{\tau_1}^{\tau_2} \tfrac{1}{2} p_1 \phi_1 \left(\frac{1}{\phi_1}\right)' dz^2 + \int_{\tau_1}^{\tau_2} \left[p_1 \left(\frac{1}{\phi_1}\right) \phi_1 z'^2 + \frac{1}{\phi_1} z (p_1 \phi_1 z')' \right] dt$$

$$= -\tfrac{1}{2} \int_{\tau_1}^{\tau_2} \left[p_1 \phi_1 \left(\frac{1}{\phi_1}\right)' \right]' z^2 dt + \int_{\tau_1}^{\tau_2} \left[p_1 z'^2 + \frac{1}{\phi_1} z (-\phi_1 q_1 z) \right] dt$$

$$= \tfrac{1}{2} \int_{\tau_1}^{\tau_2} \left[p_1 \frac{\phi_1'}{\phi_1} \right]' z^2 dt + \int_{\tau_1}^{\tau_2} (p_1 z'^2 - q_1 z^2) dt$$

Since

$$\left(\frac{p_1 \phi_1'}{\phi_1}\right)' \geq 0 \text{ on } I$$

we have

$$\int_{\tau_1}^{\tau_2} (p_1 z'^2 - q_1 z^2) dt \leq 0. \tag{26}$$

Therefore there exists a function $z(t) \in C^1[a,b]$ such that $z(a)=z(b)=0$ and satisfies the inequality (26). Using Leighton's variational theorem (Leighton 1962) we conclude that every solution $x(t)$ of (18) with $i=1$ has a zero on $[a,b]$. This is a contradiction which proves that (25) is disconjugate on I. In a similar manner we can prove that the equations

$$(\phi_i p_i y')' + \phi_i q_i y = 0, \quad i = 2, 3, \ldots, n$$

are disconjugate on I under the conditions (23) and (24). Hence (19) is disconjugate on I.

Example 5. Consider the two equations

$$x'' + e^{-(t^2/2)} \cdot (t^2/2)\, x = 0$$

$$x'' + e^{(t^2/2)}\, x = 0.$$

Sturm's comparison theorem guarantees the disconjugacy of these equations on the interval $I = [4,b]$, $(b < \infty)$. The function $\phi(t) = e^{t^2/2}$ satisfies conditions (23) and (24). The direct application of Theorem 4.2 on the above-mentioned equations shows that the equation

$$(e^{t^2/2} y')' + (t^2/4 + \tfrac{1}{2})y = 0 \qquad (*)$$

is disconjugate on I.

It is remarkable that neither Wintner's criterion (1951) nor Taam's criterion (1952) can be applied for (*).

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حول عدم ترافق المعادلة التفاضلية الخطية من المرتبة الثانية

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خلاصة

نقول بأن المعادلة (*) $(p(t)x')' + q(t)x = 0$ غير مترافقة في مجال I ، اذا كان لكل حل غير تافه صفر واحد على الأكثر في المجال I . نحصل في هذا البحث على عدد من الاختبارات لعدم ترافق المعادلة (*) . كما نبين أن اختباراتنا تنجح في كشف خاصية عدم الترافق للمعادلة (*) ، بينما تفشل بعض الاختبارات الأخرى .