*J. Univ. Kuwait (Sci.)* **3,** 1976

# Mappings and $M_i$-spaces ($i = 1, 2, 3$)

C. M. PAREEK

*Department of Mathematics, University of Kuwait*

## ABSTRACT

In this note we characterize spaces which are one to one continuous image of $M_i$-spaces and characterize those spaces which are open perfect preimage of $M_i$-spaces. Finally, we suggest several problems.

## 1. INTRODUCTION

Ceder (1961) suggested a question which is still open: does $M_3$-space imply $M_1$-space? Borges (1971) studied $M_3$-spaces where he renamed them 'stratifiable spaces'. While Borges's work did not answer Ceder's question, his theorem that closed mappings preserve stratifiable spaces did suggest new techniques. In this paper we do not obtain the solution of Ceder's problem but hope that the results will be helpful in answering it.

The following are the main results:

(a) Characterization of spaces which are one-to-one continuous image of $M_i$-spaces ($i = 1, 2, 3$).

(b) Characterization of spaces which are open perfect preimage of $M_i$-spaces.

(c) A space is $T_2$ and $F_\sigma$-screenable if and only if it is a continuous, open and $\sigma$-locally finite image of some $T_2$ paracompact space.

In this note regular space is always assumed to be $T_1$. For terms not defined in this note we refer to Engelking (1968), Ceder (1961) and Arhangel'skii (1966).

## 2. DEFINITIONS AND SOME RESULTS WHICH CAN BE PROVED EASILY FOLLOWING THE KNOWN RESULTS

*Definition 2.1.* A topological space $X$ is $F_\sigma$-screenable if every open cover of $X$ has a $\sigma$-discrete closed refinement.

*Definition 2.2.* A collection **B** of subsets of a space $X$ is called a $k$-network if for each compact set $K$ of $X$ and any open set $U$ containing $K$ there exists a finite subcollection $B_1, \ldots, B_n$ of $B$ such that $K \subset \cup^n_{i=1} B_i \subset U$.

*Definition 2.3.* Let **P** be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of a space $X$ with $P_1 \subset P_2$ for $P$ in **P**. Then:

(i) **P** is called a cushioned pair collection if for each $\mathbf{Q} \subset \mathbf{P}$ we have $cl(\cup\{P_1|P \in \mathbf{Q}\}) \subset \{P_2|P \in \mathbf{Q}\}$;

(ii) **P** is called a pair network for $X$ if for each $x$ in $X$ and any neighborhood $U$ of $x$ there is $P$ in **P** such that $x \in P_1 \subset P_2 \subset U$;

(iii) **P** is called a pair $k$-network if for each compact set $K$ of $X$ and a neighborhood $U$ of $K$ there exists a finite subcollection $P(1), \ldots, P(n)$ of $P$ such that $K \subset \cup^n{}_{i=1} P_1(i) \subset \cup^n{}_{i=1} P_2(i) \subset U$;

(iv) **P** is called a pair base for $X$ if $P$ is a pair network with the property that $P_1$ is open for each $P$ in $P$.

(v) **P** is called a pair cover of $X$ if $\cup\{P_1|P \in P\} = X$.

(vi) **P** is called an open pair cover of $X$ if for each $P$ in **P**, $P_1$ is open and **P** is a pair cover of $X$.

(vii) **P** is called a $\sigma$-cushioned (cover, open cover) if **P** is a cover (open cover) which can be expressed as a countable union of cushioned pair collection.

*Definition 2.4.* Let $(X,T)$ be a space with a map $S$ from $N \times T$ to the family of closed subsets of $X$ given, and consider the following properties that $S$ may satisfy:

(a) If $U$ is in $T$ then $U = \cup_{n=1}^{\infty} S(n,U)$.

(b) If $U$ and $V$ are in $T$ and $U \subset V$, then $S(n,U) \subset S(n,V)$ for each $n$.

(c) If $K \subset U$ where $K$ is compact and $U$ is open, then there is an $n$ such that $K \subset S(n,U)$.

(d) If $K \subset U$ where $K$ is closed and $U$ is open, then there is an $n$ such that $K \subset S(n,U)$.

Then: (i) $X$ is semi-stratifiable if $S$ satisfies (a) and (b).

(ii) $X$ is $k$-semi-stratifiable if $S$ satisfies (a), (b), and (c).

(iii) $X$ is stratifiable if $S$ satisfies (a), (b) and (d).

Definition 2.4 essentially appears in Lutzer (1971). The following theorems can be proved easily following the techniques of Siwiec & Nagata (1968).

*Theorem 2.1.* For a space $X$, the following are equivalent:

(a) $X$ is a semi-stratifiable.

(b) $X$ has a $\sigma$-cushioned pair network.

(c) For each $x$ in $X$ there exists a sequence $\{g(n,x)|n = 1, 2, \ldots\}$ of open neighborhoods of $x$ satisfying: $x$ not in $F$ where $F$ is closed, implies there is an integer $n$ such that $x \notin \cup\{g(n,y)|y \in F\}$.

*Theorem 2.2.* For a space $X$, the following are equivalent:

(i) $X$ has a $\sigma$-cushioned pair $k$-network.

(ii) $X$ is $k$-semi-stratifiable.

(iii) For each $x$ in $X$, there exists a sequence $\{g(n,x)|n = 1, 2, \ldots\}$ open neighborhoods of $x$ satisfying: $K \cap F = \varnothing$ where $F$ is closed and $K$ is compact, implies there is an integer $n$ such that $K \cap (\cup \{g(n,y)|y \in F\}) = \varnothing$.

*Theorem 2.3.* A space $X$ has a $\sigma$-closure preserving $k$-network if and only if each point $x$ of $X$ has a sequence $\{g(n,x)|n = 1, 2, \ldots\}$ of open neighborhoods satisfying:

(i) if $y \varepsilon g(n,x)$ then $g(n,y) \subset g(n,x)$ and (ii) if $K \cap F = \emptyset$ where $K$ is compact and $F$ is closed, then there is an $n$ for which $K \cap (\cup \{g(n,x) | x \varepsilon F\}) = \emptyset$.

*Question 1.* Does a space with $\sigma$-cushioned pair $k$-network have a $\sigma$-closure preserving $k$-network?

An affirmative answer to above question will give 'a first countable $M_2$-space is an $M_3$-space'.

## 3. CONTINUOUS MAPPINGS CONNECTED WITH $M_i$-SPACES ($i = 1, 2, 3$)

In this section we characterize spaces which are one-to-one continuous image of $M_i$-spaces ($i = 1, 2, 3$).

*Definition 3.1.* A function $f: X \to Y$ is called a $\sigma$-cushioned pair mapping if $f$ is such that for each $\sigma$-cushioned pair open cover $\mathbf{U}$ of $X$, $f\mathbf{U}$ can be refined by a $\sigma$-cushioned pair cover of $Y$.

*Theorem 3.1.* If $X$ is a space with $\sigma$-cushioned pair network, then $X$ has a $\sigma$-cushioned pair network $\mathbf{P} = \cup_{i=1}^{\infty} \mathbf{P}_i$ where $\mathbf{P}_i = \{(P_{\alpha 1}^i, P_{\alpha 2}^i) | \alpha \in \wedge_i\}$ for $i = 1, 2, 3, \ldots$ the following conditions:

(i) $\mathbf{P}_i$ is a cushioned pair covering of $X$ for $i = 1, 2, 3, \ldots$;

(ii) $\mathbf{P}_i \subset \mathbf{P}_{i+1}$ for $i = 1, 2, \ldots$;

(iii) $\mathbf{P}_i$ is closed under intersection for each $i$, i.e., if $\wedge_i' \subset \wedge_i$ then the pair $(\cap_{\alpha \in \wedge_i'} P_{\alpha 1}^i, \cap_{\alpha \in \wedge_i'} P_{\alpha 2}^i)$ belongs to $\mathbf{P}_i$ provided that $\cap_{\alpha \in \wedge_i'} P_{\alpha 1}^i \neq \phi$;

(iv) for each $x$ in $X$, there is a collection $\{P_\alpha^i | i = 1, 2, \ldots\}$ where $P_\alpha^i \in \mathbf{P}_i$ for $i = 1, 2, \ldots$, such that for any neighborhood $U$ of $x$ there is an $i$ such that $x$ is in $P_{\alpha 1}^i$ and $P_{\alpha 2}^i$ is contained in $U$.

*Proof.* Let $\mathbf{B} = \cup_{j=1}^{\infty} \mathbf{B}_j$ be a given $\sigma$-cushioned pair network for $X$. Let $\mathbf{B}_1 = \{(X,X)\}$. Define $\mathbf{C}_i = \bigcup_{j \leq i} \mathbf{B}_j$ and let $\mathbf{P}_i = \{(\cap_{\alpha \in F} C_{\alpha 1}^i, \cap_{\alpha \in F} C_{\alpha 2}^i) | F \subset \wedge_i$ and $\cap_{\alpha \in F} C_{\alpha 1}^i \neq \emptyset\}$ where $\mathbf{C}_i = \{(C_{\alpha 1}^i, C_{\alpha 2}^i) | \alpha \in \wedge_i\}$ for $i = 1, 2, \ldots$. We shall show that $\mathbf{P} = \cup_{i=1}^{\infty} \mathbf{P}_i$ is a required $\sigma$-cushioned pair network for $X$. $\mathbf{P}_i$ *is a cushioned pair cover of* $X$ *for each* i. Let $i$ be a fixed but an arbitrary index and let $\wedge_i' \subset 2^{\wedge_i}$ where $2^{\wedge_i}$ denotes the collection of nonempty subsets of $\wedge_i$. We wish to show that $cl(\cup \{\cap_{\alpha \in F} C_{\alpha 1}^i | F \in \wedge_i'\}) \subset \cup \{\cap_{\alpha \in F} C_{\alpha 2}^i | F \in \wedge_i'\}$. Suppose $y \notin \cup \{\cap_{\alpha \in F} C_{\alpha 2}^i | F \in \wedge_i'\}$. Then for each $F \in \wedge_i'$ there is $\alpha_F$ such that $y \notin C_{\alpha_F 2}^i$, i.e., $y \notin \{C_{\alpha_F 2}^i, F \in \wedge_i'\}$. Hence $y \notin cl(\cup \{C_{\alpha_F 1}^i | F \in \wedge_i'\})$. Therefore $y \in X - cl(\cup \{C_{\alpha_F 1}^i | F \in \wedge_i'\}) = 0_y$. Now suppose that $0_y \cap (\cup \{\cap_{\alpha \in F} C_{\alpha 1}^i | F \in \wedge_i'\}) \neq \emptyset$, i.e., there is $z \in 0_y \cap (\cup \{\cap_{\alpha \in F} C_{\alpha 1}^i | F \in \wedge_i'\})$. Consequently, for some $F_0$ in $\wedge_i'$, $z \in 0_y \cap (\cap_{\alpha \in F_0} C_{\alpha 1}^i)$. This implies $z \in 0_y$ and $z \in \cap_{\in F_0} C_{\alpha 1}^i$ which is a contradiction to the construction of $0_y$. Hence $y \in cl(\cup \{\cap_{\in F} C_{\alpha 1}^i | F \in \wedge_i'\})$. This shows that $\mathbf{P}_i$ is a cushioned pair collection for each $i$. $P_i$ is a cushioned pair cover because $(X,X) \in \mathbf{P}_i$. From the above arguments and construction of $\mathbf{P}$ it is easy to see that $\mathbf{P}$ satisfies (i) to (iii). Finally, it remains to show that $\mathbf{P}$ satisfies (iv).

For each $x$ in $X$ and each $i$, define $\wedge_x^i = \{\alpha \in \wedge_i | x \in C_{\alpha 1}^i\}$. Consider $P_i^i = (\cap_{\alpha \in \wedge_x^i} C_{\alpha 1}^i, \cap_{\alpha \in \wedge_x^i} C_{\alpha 2}^i)$ for $i = 1, 2, \ldots$. Suppose $U$ is a neighbourhood of $x$ then by the fact that $\mathbf{B}$ is a pair network there is an $i$ such that $x \in B_1^i \subset B_2^i \subset U$. Therefore by the construction $x \in \cap_{\alpha \in \wedge_x^i} C_{\alpha 1}^i \subset \cap_{\alpha \in \wedge_x^i} C_{\alpha 2}^i \subset U$. This completes the proof.

*Theorem 3.2.* For a $T_1$-space $Y$ the following statements are equivalent.

(a) $Y$ has a $\sigma$-cushioned pair network.

(b) $Y$ is the image of an $M_3$-space under a $\sigma$-cushioned pair mapping which is one to one and continuous.

*Proof.* (a) → (b). Let $\mathbf{V}$ be a $\sigma$-cushioned pair network for $Y$. Then there exist $\sigma$-cushioned pair network $\mathbf{U}$ as in theorem 3.1. Let $X$ be a copy of $Y$ topologized by taking $\mathbf{U}$ as a base for the topology of $X$. Then $X$ is a $T_1$-space with $\sigma$-cushioned pair base, i.e., $X$ is an $M_3$-space. Let $f: X \to Y$ be the identity function. Clearly, $f$ is one-to-one continuous. Also, it is easy to see that $f$ is a $\sigma$-cushioned pair mapping.

(b) → (a). The proof is evident.

*Definition 3.2.* A continuous function $f: X \to Y$ is called a *$\sigma$-closure preserving mapping* if $f$ is such that for each $\sigma$-closure preserving open cover $\mathbf{U}$ of $X$, $f(\mathbf{U})$ can be refined by a $\sigma$-closure preserving cover of $Y$.

*Theorem 3.3.* If $X$ is a topological space with $\sigma$-closure preserving closed network then $X$ has a $\sigma$-closure preserving closed network $\mathbf{P} = \cup_{i=1}^{\infty} \mathbf{P}_i$ satisfying the following properties:

(i) $\mathbf{P}_i$ is a closure preserving closed cover of $X$ for each $i$;

(ii) $P_i \subset P_{i+1}$ for each $i$;

(iii) $\mathbf{P}_i$ is closed under nonempty intersections;

(iv) for each $x$ in $X$ there exists a collection $\{P_x^i | i = 1, 2, \ldots\}$ such that $P_x^i$ belongs to $\mathbf{P}_i$ for each $i$ and for each neighborhood $U$ of $x$ there is $i$ such that $x \in P_x^i \subset U$.

The proof is similar to theorem 3.1.

*Theorem 3.4.* For a regular space $Y$ the following are equivalent:

(a) $Y$ has a $\sigma$-closure preserving network.

(b) $Y$ is an image of an $M_1$-space under a $\sigma$-closure preserving mapping which is one to one and continuous.

The proof is similar to theorem 3.2.

In the remaining part of this section we shall give a partial solution of a problem of Michael (1970).

*Definition 3.3.* A continuous function $f: X \to Y$ is called a $\sigma$-locally finite mapping if for every $\sigma$-locally finite open cover $\mathbf{U}$ of $X$, $f(\mathbf{U})$ can be refined by a $\sigma$-locally finite cover of $Y$.

*Lemma 3.5.* Every space $X$ is an open quotient image of a paracompact space $Y$ in which for every family of open sets there is a disjoint family of open sets having the same union. Also, if $X$ is $T_2$ then $Y$ is $T_2$.

*Lemma 3.6.* If $f$ is a continuous open mapping of a space $X$ onto a $F_\sigma$-screenable space $Y$ then $f$ is a $\sigma$-locally finite mapping.

The proof is a trivial consequence of the definition of $F_\sigma$-screenable spaces.

*Theorem 3.7.* A space $Y$ is $T_2$ and $F_\sigma$-screenable if and only if it is an image of a paracompact $T_2$ space under an open $\sigma$-locally finite mapping.*

The proof of the theorem follows from lemmas 3.6 and 3.7.

* By Hanai (1961) open maps are not $\sigma$-locally finite in general.

## 4. DECOMPOSITION MOD BASIS, DECOMPOSITION MOD NETWORK AND $M_i$-SPACES

*Definition 4.1.* Let X be a set. A partition of $X$ is defined to be a disjoint family **A** of non-empty subsets of $X$ such that $X = \cup\{A|A \in \mathbf{A}\}$. If $X$ is a space then a partition **A** of $X$ is called a decomposition of $X$ if each member of **A** is a compact subset of $X$. A decomposition **A** of a topological space $X$ is called a closed decomposition (or an open decomposition) if for each closed set $C$ (or each open set $U$) the set $C^{\#} = \cup\{A|A \cap C \neq \varnothing, A \varepsilon \mathbf{A}\}$, (or $\mathbf{U}^{\#} = \cup\{A|A \cap U \neq \varnothing, A \in \mathbf{A}\}$) is closed (or open) in $X$. A decomposition is called a closed and open decomposition if it is a closed and open decomposition simultaneously.

*Definition 4.2.* Let **A** be a cover (not necessarily open) of a space $X$. Then **A** is a network for X if for each neighborhood $U$ of $x$ there is $A$ in **A** such that $x$ is in $A$ and $A$ is contained in $U$. Analogously, call **A** a decomposition (mod $k$)-network (quasi-decomposition (mod $k$)-network) for $X$ if there exists a decomposition **K** of $X$ such that whenever $K \subset U$ with $K$ in **K** and $U$ open in $X$ then $K \subset A \subset U$ ($K \subset$ int $A \subset U$) for some $A$ in **A**. A network the interior of whose elements form a base is called a quasi-base and a network whose elements are open sets is of course a base. Therefore, by analogy call a decomposition (mod $k$)-network whose elements are open sets a decomposition (mod $k$)-base, and quasi-decomposition (mod $k$)-network a decomposition (mod $k$)-quasi-base. Now, one can make obvious definitions for $\sigma$-locally finite, $\sigma$-closure preserving, $\sigma$-cushioned pair, and $\sigma$-quasi decomposition (mod $k$)-base respectively. Also, similar notions for networks can be defined.

*Theorem 4.1.* The following properties for a regular space $X$ are equivalent:

(a) $X$ has a $\sigma$-locally finite closed and open decomposition (mod $k$)-base.

(b) There exists a perfect open map from $X$ onto a metric space.

*Proof.* (a)→(b). Let **K** be an open and closed decomposition for $X$ and let **B** be a $\sigma$-locally finite (mod $k$)-base. $K$ induces a natural equivalence relation $R$ on $X$. The quotient space $X/R$ can be shown to be metrizable and the quotient map $\varnothing: \mathrm{X} \to X/R$ can be shown to be perfect and open.

(b)→(a). Let $X$ be a regular space and let $f: \mathrm{X} \to Y$ be a perfect open mapping of a space $X$ onto a metric space $Y$. Let $\mathbf{B} = \cup_{i=1}^{\infty} \mathbf{B}_i$ be a $\sigma$-locally finite base for $Y$. Then $\mathbf{K} = \{f^{-1}y|y \in Y\}$ is a decomposition of $X$. That **K** is open and closed follows from the fact that $f$ is continuous, open and closed. Since $f$ is continuous and closed one can easily show that $f^{-1}(\mathbf{B})$ is a $\sigma$-locally finite decomposition (mod $k$)-base. Hence the theorem is proved.

Now, we state without proof some analogues of the above theorem.

*Theorem 4.2.* The following statements for a regular space $X$ are equivalent:

(a) $X$ has a $\sigma$-closure preserving (or $\sigma$-cushioned pair) closed and open decomposition (mod $k$)-base.

(b) There exists a perfect open map from $X$ onto an $M_1$-space (or $M_3$-space).

*Theorem 4.3.* The following statements for a regular space $X$ are equivalent:

(a) $X$ has a $\sigma$-closure preserving closed and open decomposition (mod $k$)-quasi-base.

(b) There exists a perfect open map from $X$ onto an $M_2$-space.

*Theorem 4.4.* The following statements for a regular space $X$ are equivalent:

(a) $X$ has a $\sigma$-discrete, closed decomposition (mod $k$)-network.

(b) $X$ has a $\sigma$-locally finite, closed decomposition (mod $k$)-network.

(c) $X$ has a $\sigma$-closure preserving, closed decomposition (mod $k$)-network.

(d) There exists a perfect mapping of $X$ onto a $\sigma$-space.

The proof follows easily from Siwiec & Nagata (1968).

## 5. PROBLEMS

*5.1.* Is a regular first countable space with a $\sigma$-closure preserving $k$-network an $M_1$-space?

*5.2.* Does there exist a regular space with a $\sigma$-cushioned pair $k$-network which does not have a $\sigma$-closure preserving $k$-network?

*5.3.* Is it true that for a regular space $X$ the following are equivalent?

(a) $X$ has a $\sigma$-closure preserving closed decomposition (mod $k$)-base.

(b) $X$ has a $\sigma$-closure preserving, closed decomposition (mod $k$)-quasi-base.

(c) $X$ has a $\sigma$-cushioned pair, closed decomposition (mod $k$)-base.

## REFERENCES

**Arhangel'skii, A.V. 1966.** Mappings and spaces, Russian Math. Surveys **21:** 115–62.

**Borges, C.R. 1971.** A survey of $M_1$-spaces: open questions and partial results. General Topology and its Applications **1:** 79–84.

**Ceder, J.R. 1961.** Some generalizations of metric spaces. Pacific J. Math. **11:** 105–25.

**Engelking, R. 1968.** Outline of general topology. North-Holland, Amsterdam.

**Hanai, S. 1961.** On open mappings II. Proc. Japan Acad. **37:** 233–8.

**Lutzer, D.J. 1971.** Semimetrizable and semistratifiable spaces, General Topology and its Applications **1:** 43–8.

**Michael, E.A. 1970.** On Nagami's Σ-spaces and some related matters. Proc. Washington State Univ. Conference on general topology: 13–19.

**Siwiec, F. & Nagata, J. 1968.** A note on nets and metrization, Proc. Japan Acad. **44:** 623–7.

(*Received* 22 *September* 1976)

# دوال وفضاءات م هـ (هـ = ١ ، ٢ ، ٣)

س . م . باريك
قسم الرياضيات بجامعة الكويت

## خلاصـــة

يتوصل المؤلف ، فى هذا البحث ، إلى تعيين الفضاءات التى هى صور فضاءات م هـ المتصلة تقابلا وكذلك الفضاءات ما قبل صور م هـ التامة المفتوحة . ثم يقترح عدة مسائل .