J. Univ. Kuwait (Sci.) **13,** 1986

On the zeros of a polynomial and its derivative. I.

Q. M. TARIQ

Department of Mathematics, Indian Institute of Technology, Kanpur 208 016, India

ABSTRACT

Let $p(z)$ be a polynomial having all its zeros in the closed unit disk. Given that a is a zero of multiplicity k we seek to determine the smallest disk centred at a containing at least k zeros of the derivative $p'(z)$.

INTRODUCTION

We denote by $D(a; R)$ the open disk $\{z: |z - a| < R\}$ and by $\overline{D(a; R)}$ its closure. Let $p(z)$ be a polynomial having all its zeros in $\overline{D(0; 1)}$. It has been conjectured by Sendov (see Marden 1983) that for each zero a of $p(z)$ the disk $\overline{D(a; 1)}$ contains at least one zero of the derivative $p'(z)$. This is, in fact, obvious if a happens to be a multiple zero, since then $p'(a) = 0$. A non-trivial question in the case of a zero a of multiplicity k is to ask for the smallest number R_k such that the disk $\overline{D(a; R_k)}$ contains at least k zeros of $p'(z)$. This is clearly a harder problem than the one of Sendov. Here we only consider the case of a boundary zero of multiplicity k and obtain the following extension of a result of Goodman *et al.* (1969), Schmeisser (1969), and Meir & Sharma (1969).

Theorem. Let $|a| = 1$. If $p(z) := c(z - a)^k \prod_{j=1}^{n-k} (z - z_j)$ is a polynomial of degree n $(> k)$ such that $|z_j| \leqslant 1$ for $j = 1, \ldots, n - k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $\overline{D\left(\frac{a}{k+1}; \frac{k}{k+1}\right)}$.

PROOF OF THE THEOREM

Let us write

$$p(z) = (z - a)^k q(z) \quad . \tag{1}$$

Without loss of generality we may assume $q(a) \neq 0$. Let us denote by $w_1, \ldots, w_{n-k}$ the zeros of $p'(z)$ other than the $(k - 1)$-fold zero at a. Since

$$\frac{p''(z)}{p'(z)} = \frac{k-1}{z-a} + \sum_{j=1}^{n-k} \frac{1}{z - w_j}$$

the function

$$f(z) := \sum_{j=1}^{n-k} \frac{1}{z - w_j} = \frac{p''(z)}{p'(z)} - \frac{k-1}{z-a} = \frac{(z-a)p''(z) - (k-1)p'(z)}{(z-a)p'(z)} \tag{2}$$

is holomorphic in a neighbourhood of the point a. From (1) we have

$$p'(z) = (z-a)^k q'(z) + k(z-a)^{k-1} q(z) \quad ,$$

$$p''(z) = (z-a)^k q''(z) + 2k(z-a)^{k-1} q'(z) + k(k-1)(z-a)^{k-2} q(z) \quad ,$$

so that

$$f(z) = \frac{(z-a)q''(z) + (k+1)q'(z)}{(z-a)q'(z) + kq(z)} \quad .$$

In particular, we obtain

$$\sum_{j=1}^{n-k} \frac{1}{a - w_j} = f(a) = \frac{k+1}{k} \frac{q'(a)}{q(a)} \quad . \tag{3}$$

Since $q(z) := c \prod_{j=1}^{n-k} (z - z_j)$, we have

$$\frac{q'(a)}{q(a)} = \sum_{j=1}^{n-k} \frac{1}{a - z_j}$$

and so

$$\sum_{j=1}^{n-k} \frac{a}{a - w_j} = \frac{k+1}{k} \sum_{j=1}^{n-k} \frac{a}{a - z_j} \quad . \tag{4}$$

Taking real parts of the two sides of (4), we get

$$\sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - w_j} = \frac{k+1}{k} \sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - z_j} \quad . \tag{5}$$

The hypothesis $|z_j| \leqslant 1$ implies that $\operatorname{Re} \dfrac{a}{a - z_j} \geqslant \dfrac{1}{2}$ for $1 \leqslant j \leqslant n - k$. Thus

$$\sum_{j=1}^{n-k} \operatorname{Re} \frac{a}{a - w_j} \geqslant \frac{k+1}{k} \frac{n-k}{2}$$

and $\operatorname{Re} \dfrac{a}{a - w_j} \geqslant \dfrac{k+1}{2k}$ for some $j(1 \leqslant j \leqslant n - k)$. This proves that $p'(z)$ has at least one zero $\neq a$ in $\overline{D\left(\dfrac{a}{k+1}; \dfrac{k}{k+1}\right)}$. Since it has a $(k-1)$-fold zero at a the theorem follows.

AN EXAMPLE

Let α_k be the unique number in $(\pi/2, \pi)$ such that

$$\cos \alpha_k = -\frac{(k+1)^2 - 2}{(k+1)^2}$$

and consider the particular polynomial

$$p(z) := (z - 1)^k (z^2 - 2z \cos \alpha + 1) \quad .$$

Then, in addition to a $(k - 1)$-fold zero at 1, $p'(z)$ has zeros at

$$w_1, w_2 := \frac{1 + (k+1)\cos\alpha \pm i\sqrt{(k+2)(k + 2\cos\alpha) - \{1 + (k+1)\cos\alpha\}^2}}{k+2}$$

where the quantity under the radical sign is positive if $0 < \alpha < \alpha_k$. It is easily checked that as α runs from 0 to α_k the points w_1 and w_2 describe the boundary of $\overline{D\left(\frac{1}{k+1}; \frac{k}{k+1}\right)}$. This proves the sharpness of our result.

A COROLLARY

As an immediate consequence of our theorem we obtain

Corollary. Under the assumptions of the theorem, $p'(z)$ has at least k zeros in $\overline{D\left(a; \frac{2k}{k+1}\right)}$.

If $p(z) := (z + 1)(z - 1)^k$, then $p'(z)$ has a $(k - 1)$-fold zero at 1 and a simple zero at $-\frac{k-1}{k+1}$. This shows that for every $\varrho < \frac{2k}{k+1}$ the disk $\overline{D(a; \varrho)}$ may not contain more than $k - 1$ zeros of $p'(z)$. As another example we may consider the polynomial $p(z) := \left(z^2 + 2\frac{(k+1)^2 - 2}{(k+1)^2} z + 1\right)(z - 1)^k$ whose derivative has a double zero at $-\frac{k-1}{k+1}$ in addition to a $(k - 1)$-fold zero at 1.

A REMARK

Note that k is allowed to be 1 in the theorem as well as in the corollary.

REFERENCES

Goodman, A.W., Rahman, Q.I. & Ratti, J.S. 1969. On the zeros of a polynomial and its derivative. Proceedings of the American Mathematical Society **21**: 273–74.

Marden, M. 1983. Conjectures on the critical points of a polynomial. American Mathematical Monthly **90**: 267–76.

Meir, A. & Sharma, A. 1969. On Ilyeff's conjecture. Pacific Journal of Mathematics **31**: 459–67.

Schmeisser, G. 1969. Bemerkungen zu einer Vermutung von Ilieff. Mathematische Zeitschrift **111**: 121–25.

(*Received 19 February 1985, revised 23 June 1985*)

حول اصفار الحدودية ومشتقاتها

ك . م . طارق
قسم الرياضيات بالمعهد الهندي للتكنولوجيا ، كانپور ، الهند

خلاصة

لنفرض $p(z)$ حدودية تقع جميع اصفارها في قرص الوحدة المغلق . فاذا كانت a صفرا ذا تردد k ، فان المؤلف يبحث عن اصغر قرص مركزه a ويحتوي على k من اصفار المشتقة $p'(z)$ على الاقل .