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Bases and maximal ideals in semigroups

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ABSTRACT

The notion of B-semigroup is defined in this paper, and some properties of B-semigroups are studied. The relation between maximal ideals and bases is discussed. Also some results about B-semigroups and about the relation between one-sided and two-sided bases are obtained.

1. INTRODUCTION

Tamura (1955) introduced the notion of one-sided bases of semigroups to show that all one-sided translations of a semigroup having one-sided base can be constructed somewhat easily. Al-Lahham (1974) introduced the notion of basic elements of semigroups, one-sided and two-sided, which can be considered as a special case of bases in general, since they are bases with only one element.

Fabrici (1975) introduced the notion of two-sided bases of semigroups and investigated some properties of semigroups with one-sided and two-sided bases.

In this paper we shall give the definition of bases in its general case, and introduce the notion of B-semigroups; also we shall discuss the relation between maximal ideals and bases in B-semigroups. We shall follow the notation and terminology of Clifford & Preston (1961, 1967) for all concepts not defined in this paper.

Propositions in this paper will be mentioned for the right case and will be omitted for the left case and the two-sided case.

2. BASIC DEFINITIONS

A subset A of a semigroup S is called a right (left, two-sided) base of S if:

(1) $A\cup SA=S$ $(A\cup AS=S,\ A\cup SA\cup AS\cup SAS=S)$.
(2) No proper subset D of A exists in S for which $D\cup SD=S$ $(D\cup DS=S,$ $D\cup SD\cup DS\cup SDS=S)$.

A semigroup S with at least one one-sided or two-sided base is called a B-semigroup.

Note. B-semigroup in the following means a semigroup with at least one right base.

A principal left (right, two-sided) ideal of a semigroup S generated by an element

$a \in S$ will be denoted by $L(a)$ ($R(a)$, $J(a)$). The set of all elements of a semigroup S which are $\mathcal{L}$-equivalent to an element $a \in S$ will be denoted by $L_a = \{x \in S: L(x) = L(a)\}$, similarly for R_a and J_a. A semigroup S is called normal if $xS = Sx$ for every x in S. An element a in a semigroup S is called a right (left, two-sided) basic element of S if $\{a\} \cup Sa = S$ ($\{a\} \cup aS = S$, $\{a\} \cup aS \cup Sa \cup SaS = S$).

3. PRINCIPAL LEMMAS

Let us define a quasi-ordering into a semigroup S:

$$a \leq b \text{ means } L(a) \subseteq L(b) \; a,b \in S$$

An element a of a semigroup S is called a maximal left (right, two-sided) element in S if $L(a)$ ($R(a)$, $J(a)$) is not properly contained in any $L(x)$ ($R(x)$, $J(x)$) for every x in S.

It is well known (Tamura 1955) that if S is a semigroup and A is a right base of S, then

Lemma 1. If a,b in A and a in Sb, then $a = b$.

Lemma 2. If $a \neq b$ where $a,b \in A$, then $a \not\leq b$ as well as $a \not\geq b$.

Lemma 3. A non-empty subset A of a semigroup S is a right base of S, iff A satisfies the following:

(1) For every $x \in S$ there exists $a \in A$ such that $x \leq a$.
(2) Neither $a \leq b$ nor $a \geq b$ for any two distinct elements $a,b \in A$.

Example. A semigroup can be without any base at all, neither one-sided nor two-sided. The following semigroup illustrates this. Let S be the semigroup of all positive integers with the binary operation $xy = \min(x,y)$ $x,y \in S$. Then for every $x \in S$

$$L(x) = \{1,2,\ldots,x\} \text{ and } L_x = \{x\} \quad .$$

Therefore $L(1) \subset L(2) \subset L(3) \subset \ldots \subset L(x) \subset \ldots$ and S has no right bases. But S is a commutative semigroup, so S has neither left nor two-sided bases.

Remark. In these preceding example $|L_x| = 1$ for every x in S. A question arises here: Is it true that any semigroup S has no right base if $|L_x| = 1$ for every x in S? This does not hold as the following example shows. Let $S = \{(s,t): s,t \in N\}$ with the binary operation $(s,t)(s',t') = (ss', ts' + t')$ where $N = \{1,2,3,\ldots\}$. Then S is a semigroup with $|L_x| = 1$ for every x in S. To prove that, let us suppose, as a way of contradiction, that there exists $y \neq x$ in L_x, that is, $x \in Sy$ and $y \in Sx$. If $x = (s, t)$, and $y = (s',t')$, then there exists (a,b), (a',b') in S such that $(s,t) = (a,b)(s',t')$; $(s',t') = (a',b')(s,t)$, so $t = bs' + t'$ and $t' = b's + t$. It follows that $bs' + b's = 0$, which contradicts our hypothesis that b,s',b',s are natural numbers. Now $S(1,1) \cup (1,1) = S$. Therefore S has a right base $A = \{(1,1)\}$.

4. THE MAIN RESULTS

Theorem 1. A semigroup S has a right base iff for every $x \in S$ there exists a maximal left element $a \in S$ such that $x \leq a$. Moreover, in a B-semigroup every right base is a subset of the set of all maximal left elements, and every maximal left element belongs to some right base of S.

Proof. The proof is obvious.

Example. Let S be the multiplicative semigroup of all real numbers from the half-open interval $[0,1)$, then for every two distinct elements $x,y \in [0,1]$ we have $L(x) \neq L(y)$ since $L(x)=[0,x]$ and $L(y)=[0,y]$. If $y>x$ then $L(y) \supset L(x)$, that is, S has no maximal left elements; therefore S has no right base. But S is commutative, so it has neither left nor two-sided bases.

Remark. It is worthwhile to observe that the existence of maximal left elements is not enough for a semigroup to have a right base. The following counterexample illustrates that.

Let $S=[0,1) \cup \{a,b,c,d\}$ with the binary operation defined as follows:

$x*y = 0$ if x and y are not in the same set
$x*y$ has its value from Table 1 when $x,y \in \{a,b,c,d\}$
$x*y$ has its multiplicative value when $x,y \in [0,1)$.

Then S is a semigroup with two maximal left elements b and d, but it has no right base.

Table 1

	a	*b*	*c*	*d*
a	a	a	a	a
b	a	b	a	d
c	c	c	c	c
d	a	b	a	d

Corollary. Every finite semigroup has a right base, so every finite semigroup is a B-semigroup.

Theorem 2. (i) A B-semigroup S has more than one right base iff a right base of S has at least one element a, such that $|L_a|>1$.

(ii) A finite semigroup S with a right base $A=\{a_1,a_2,\ldots,a_n\}$ has $|L_{a_1}| \cdot |L_{a_2}| \cdot \ldots \cdot |L_{a_n}|$ right bases.

Proof. The proof is obvious.

Example. Let S be the multiplicative semigroup of the natural numbers $S=N-\{1\}=\{2,3,\ldots\}$. Then for every $x \in S$: $L(x)=\{nx : n \in N\}$. Let a,b; $(a<b)$ be any two elements in S, then $L(a) \neq L(b)$ since $a \notin L(b)$ and $a \in L(a)$. So $|L_x|=1$ for every $x \in S$.

Now every prime number $p \in S$ is a maximal left element in S since $L(x) \supseteq L(p)$ implies that x divides p; hence $x=p$, and $L(p)$ is not properly contained in any $L(x)$, $x \in S$.

Let $P=\{p \in S : p$ is a prime number$\}$. For every $x \in S$ there exists $p \in P$ such that p

divides x. So for every $x \in S$ there exists $p \in P$ such that $L(p) \supseteq L(x)$, that is $p \geqslant x$. Hence S has a right base $A \subseteq P$. But $|L_a| = 1$ for every $a \in A$. Therefore, A is unique and $A = P$.

Theorem 3. If A and A' are two distinct right bases of a B-semigroup S, then

(1) For every $a \in A$ there exists a unique $b \in A'$ such that $L_a = L_b$.
(2) There exists a bijection ϕ: $A \to A'$; $a \to b$, where $L_a = L_b$.
(3) A and A' have the same cardinality.

Proof. The proof is obvious.

Theorem 4. In a semigroup S, $|L_x| > 1$ for some element $x \in S$ iff there exist a and b in S such that $x = abx$ with $bx \neq x$.

Proof. If $|L_x| > 1$ for some element $x \in S$, then there exists $y \neq x$ in S such that $L(y) = L(x)$, so there exist $a,b \in S$ such that $x = ay$ and $y = bx$. Hence $x = abx$ with $bx \neq x$.

If $a,b,x \in S$ such that $x = abx$ with $bx \neq x$, then $L(bx) \subseteq L(x)$. But $x = abx$ implies that $L(x) \subseteq (bx)$. Therefore, $L(x) = L(bx)$, $x \neq bx$ and $|L_x| > 1$.

Theorem 5. If A is a right base of a B-semigroup S, then the following are mutually equivalent:

(1) S has more than one right base.
(2) There exists $a \in A$ such that $|L_a| > 1$.
(3) There exist $a \in A$ and $x,y \in S$ such that $a = xya$ and $ya \neq a$.

Proof. The proof is obvious.

Theorem 6. If A is the unique right base of a B-semigroup S, then the complement $\bar{A}$ of A is a sub-semigroup of S.

Proof. Let $a,b \in \bar{A}$. Since $ab \in Sb$ then $ab \leq b$. But $b \notin A$ implies that there exists $c \in A$ such that $b \leq c$, by Lemma 3, so $ab \leq b \leq c$.

If $ab \in A$ then $ab = c$ and $L(ab) \subseteq L(b) \subseteq L(c) = L(ab)$. Then $L(ab) = L(b)$, and $b \in L_c$, where $b \neq c$. So $|L_c| > 1$, which contradicts our hypothesis that A is unique. Then $ab \notin A$, that is, $ab \in \bar{A}$ and $\bar{A}$ is a sub-semigroup of S.

Example. let S be the semigroup defined by Table 2.

Table 2

	a	b	c	d	e	f
a	b	b	e	e	e	b
b	b	b	e	e	e	b
c	f	f	d	d	d	f
d	f	f	d	d	d	f
e	b	b	e	e	e	b
f	f	f	d	d	d	f

Then $A=\{a,c\}$ is the unique right base of S, and $\bar{A}=\{b,d,e,f\}$ is a sub-semigroup of S.

Remark. The converse of the last theorem is not true. For a counterexample let S be the semigroup defined by Table 3.

Then $A=\{c\}$ is a right base of S, and $\bar{A}=\{a,b,d,e\}$ is a sub-semigroup of S. But A is not unique since $B=\{b\}$ is another right base of S.

Table 3

	a	*b*	*c*	*d*	*e*
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

Theorem 7. If A is a right base of a B-semigroup S, then A is unique iff $\bar{A}$ is a left ideal of S.

Proof. (i) A is unique.

Let x be any element of S, and b be any element of $\bar{A}$, then $xb\in Sb$ and $xb\leq b$. $b\notin A$ implies that there exists $c\in A$ such that $b\leq c$, that is $xb\leq b\leq c$. If $xb\in A$ then $xb=c$ and $b\in L_c$, with $b\neq c$, then $|L_c|>1$, which contradicts the hypothesis that A is unique. Therefore $xb\notin A$ and $\bar{A}$ is a left ideal of S.

(ii) $\bar{A}$ is a left ideal of S. Let us suppose that there exists an element $a\in A$ such that $|L_a|>1$, which implies that there exists an element $b\in\bar{A}$ such that $b\in L_{\hat{a}}$. But $b\in\bar{A}$ implies that $L(b)\subseteq\bar{A}$, since $\bar{A}$ is a left ideal. So $L(a)\subseteq\bar{A}$, that is $a\in\bar{A}$, contrary to the hypothesis that $a\in A$. Therefore, for every $a\in A$, $|L_a|=1$ and A is the unique right base of S.

Theorem 8. Let A be a right base of a B-semigroup S, then the following are mutually equivalent;

(1) A is unique.
(2) For every $a\in A$, $|L_a|=1$.
(3) $\bar{A}$ is a left ideal of S.

Proof. The proof is obvious.

Theorem 9. If A is a right base of a B-semigroup S, then a left ideal M of S is maximal iff $S-M=L_a$ for some element $a\in A$.

Proof. (i) $S-M=L_a$ for such an element $a\in A$. Since $L_a\subseteq L(a)$ then $M\cup L(a)=S$. Let K be a left ideal of S which properly contains M, then $x\in K-M$ implies $x\in L_a$. But $M\cup L(x)\subseteq K$, so $M\cup L(x)\subseteq K$, that is, $S=K$. Therefore, M is a maximal left ideal of S.

(ii) M is a maximal left ideal of S. A cannot be a subset of M because $A\subseteq M$ implies that $S=SA\cup A\subseteq SM\cup M\subseteq M\subseteq S$. So $M=S$, which contradicts the hypothesis that M is a maximal left ideal of S. Then $\bar{M}\cap A\neq\emptyset$.

Let $a \in \bar{M} \cap A$ then $M \cup L(a) = S$, since M is maximal, that is, $\bar{M} \subseteq L(a)$. So for every $x \in \bar{M}$, $L(x) \subseteq L(a)$, $x \neq a$. So $x \leq a$ and $x \notin A$ by Lemma 3. Therefore $\bar{M} \cap A = \{a\}$.

Now $L_a \cap M = \emptyset$ because $b \in L_a \cap M$ implies that $L(b) \subseteq M$, that is, $L(a) \subseteq M$ which implies that $a \in M$, but $a \in \bar{M}$, so $L_a \subseteq \bar{M}$. But $M \cup L(x) = S$ for every $x \in M$ because M is maximal, so $\bar{M} \subseteq L(x)$. It follows that $a \in L(x)$ and $L(a) \subseteq L(x)$. Hence $L(x) = L(a)$ and $x \in L_a$, that is, $\bar{M} \subseteq L_a$. Therefore, $\bar{M} = L_a$.

Corollary 1. Let $\mathscr{M}$ be the set of all maximal left ideals of a B-semigroup S and let A be a right base of S, then

(1) $\mathscr{M}$ is not empty.
(2) There exists a bijection ϕ: $A \to \mathscr{M}$: $a \to M_a$ where $M_a = S - L_a$.
(3) $\mathscr{M}$ has the same cardinality as A.
(4) Every maximal left ideal of S is the complement of an $\mathscr{L}$-class of an element of A and vice versa.

Corollary 2. A semigroup S has a universally maximal left ideal L^* iff it has a right base A such that $|A| = 1$ and $a \in A$ implies that $L_a \neq S$. So we have proved the following theorem (Al-Lahham 1974).

A semigroup S has a universally maximal left (right, two-sided) ideal $L^*(R^*, T^*)$ iff S has at least one—but not all—right (left, two-sided) basic element.

Theorem 10. If $A = \{a\}$ is a right base of a B-semigroup S and $a \notin Sa$, then A is the unique right base of S and A is the universally maximal left ideal of S.

Proof. $\bar{A} = S - A = Sa$, so $\bar{A}$ is a left ideal of S. A is unique, by Theorem 7, so $\bar{A}$ is *a* universally maximal left ideal.

Example. Let S be the semigroup of all positive integers with addition, then $A = \{1\}$ is a right base of S and $1 \notin S + 1$, so A is unique and $\bar{A} = \{2,3,4, \ldots\}$ is the universally maximal left ideal of S.

Theorem 11. If a is an element of a B-semigroup S, then $S - \{a\}$ contains a maximal left ideal of S iff a belongs to some right base A of S.

Proof. If a belongs to some right base A of S, then $M_a = S - L_a$ is a maximal left ideal of S, by Theorem 9. But $S - L_a \subseteq S - \{a\}$, so $S - \{a\}$ contains the maximal left ideal M_a. If $S - \{a\}$ contains a maximal left ideal M of S, then the complement $\bar{M}$ of M is equal to some $\mathscr{L}$-class L_b for some $b \in A$, by Theorem 9. Hence $a \in L_b$, that is, a belongs to some right base of S.

It seems from Theorems 9 and 11 that the presence of maximal ideals in a semigroup is very close to the existence of bases in this semigroup. The following counterexample explains the falsity of the observation.

Example. Let S be the 0-direct union of two semigroups S_1 and S_2 where S_1 is the multiplicative semigroup of rational numbers from the open interval $(-1, +1)$, and S_2 is the semigroup $\{0,a\}$ where $a^2 = a$ and the element 0 has the usual property of the multiplicative zero.

Then

$$L(x) = \{y \in S: -|x| < y < |x|\} \cup \{x\} \text{ when } x \in S_1$$
$$L(a) = \{0, a\}$$
$$L(0) = \{0\}$$

Then a is the unique maximal left element in S. But $x \not\leq a$ for every $x \in S_1 - \{0\}$. So S has no right base and S has a maximal left ideal S_1.

Note. S_1 is the unique maximal left ideal of S, but it is not the universally maximal left ideal of S since S_2 is a left ideal of S and $S_2 \nsubseteq S_1$.

Theorem 12. In a normal B-semigroup, every right base is a left base and vice versa. Moreover, every right base is a two-sided base.

Proof. let A be a right base of a B-semigroup S. If S is normal, then $S = SA \cup A = AS \cup A$. Let D be a proper subset of A such that $S = DS \cup D$, then $S = SD \cup D$; but A is a right base, so A is minimal and $S = SD \cup D$ is impossible. So A is a left base of S.

Now $S = SA \cup A$ implies that $S = SA \cup AS \cup SAS \cup A$. Let B be a proper subset of A such that $S = SB \cup BS \cup SBS \cup B$, then $S \subseteq B \cup SB \subseteq S$. Hence $S = B \cup SB$, contrary to the hypothesis that A is a right base of S. Therefore, A is a two-sided base of S.

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القواعد والمثل العظمى في أنصاف الزمر

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قسم الرياضيات بجامعة دمشق

خلاصة

ان هذا البحث يتضمن تعريف مفهوم نصف الزمرة القاعدية ، ودراسة بعض مميزاتها وخصائصها ، كما يبحث في تعيين الشروط اللازمة والكافية لتكون القاعدة وحيدة أو متعددة في نصف زمرة قاعدية . ويتطرق البحث لدراسة العلاقة بين المثل العظمى والقواعد في نصف زمرة قاعدية ، بالاضافة إلى ايجاد بعض الشروط اللازمة وغير الكافية لقاعدة أحادية الجانب كي تكون ثنائية الجانب .