J. Univ. Kuwait (Sci.) **12,** 1985

On the reducibility of the triple hypergeometric function H_A

M. I. QURESHI, F. U. KHAN AND M. A. PATHAN

Department of Mathematics, Aligarh Muslim University, Aligarh, U.P., India

ABSTRACT

In this paper, certain reduction formulae for the triple hypergeometric function H_A are obtained.

1. INTRODUCTION

As usual, let

$$(\alpha)_j = \Gamma(\alpha+j)/\Gamma(\alpha) = \alpha(\alpha+1)\ldots(\alpha+j-1) \quad \text{for } j>0, \quad (\alpha)_0 = 1$$

and

$$(\alpha)_j = (-1)^j/(1-\alpha)_{-j} \quad \text{for } j<0 \quad .$$

The function H_A is defined (Srivastava 1964) by the triple series

$$H_A[a,b,c;d,e;x,y,z] = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+p}(b)_{m+n}(c)_{n+p}}{(d)_m(e)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!} \quad , \tag{1.1}$$

whose region of convergence is given by

$$|x|<r, \quad |y|<s, \quad |z|<t, \quad r+s+t = 1+st \quad .$$

The two main reduction formulae, namely (2.2) and (2.3) of Srivastava (1964) are in terms of the Horn's double hypergeometric functions (Exton 1976, p. 36)

$$G_2[a,b,c,d;x,y] = \sum_{m,n=0}^{\infty} (a)_m(b)_n(c)_{n-m}(d)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$H_2[a,b,c,d;e;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m}\frac{x^m}{m!}\frac{y^n}{n!} \quad .$$

For the benefit of the reader we recall the definitions of the Appell's hypergeometric functions (Bailey 1935, p. 73; Exton 1976, p. 23)

$$F_2[a;\, b,\, c;\, d,\, e;\, x,\, y] \;=\; \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!} \quad ,$$

$$F_4[a;\, b;\, c,\, d;\, x,\, y] \;=\; \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}x^m y^n}{(c)_m(d)_n m!\, n!} \quad ,$$

and also of the Gauss's function (Bailey 1935, p. 1; Exton 1976, p. 13)

$${}_2F_1[a,\, b;\, c;\, x] \;=\; \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \quad .$$

It may be mentioned that $F_2[a;\, b,\, c;\, d,\, e;\, x,\, y]$ reduces to the function ${}_2F_1[a,\, b;\, d;\, x]$ when c is zero.

We will need to refer to the following transformations involving the functions F_2, G_2 and H_2 (Erdélyi 1950, pp. 149 (21), 150 (22); Exton 1976, p. 37 (1.6.1.4, 1.6.1.5)):

$$G_2[a,\, b,\, c,\, d;\, x,\, y]$$

$$= (1 + x)^{-a}(1 + y)^{-b}F_2\left[1 - c - d;\, a,\, b;\, 1 - c,\, 1 - d;\, \frac{x}{x + 1},\, \frac{y}{y + 1}\right] \quad , \tag{1.2}$$

$$G_2[a,\, b,\, c,\, d;\, x,\, y]$$

$$= (1 + y)^{-b}H_2\left[d,\, a,\, b,\, 1 - c - d;\, 1 - c;\, -x,\, \frac{-y}{y + 1}\right] \quad . \tag{1.3}$$

2. THE REDUCTION FORMULAE

2.1. It will be always assumed that the appropriate convergence conditions are satisfied. By elementary manipulation of the series involved, we obtain

$$H_A[a + d - 1,\, b,\, c;\, a,\, c;\, x,\, y,\, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(a + d - 1)_{m+p}(b)_{m+n}(c)_{n+p}}{(a)_m(c)_{n+p}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}$$

$$= \sum_{m,p=0}^{\infty} \frac{(a + d - 1)_{m+p}(b)_m}{(a)_m}\frac{x^m}{m!}\frac{z^p}{p!}\sum_{n=0}^{\infty}(b + m)_n\frac{y^n}{n!}$$

$$= (1 - y)^{-b}\sum_{m,p=0}^{\infty} \frac{(a + d - 1)_{m+p}(b)_m(e)_p}{(a)_m(e)_p}\frac{1}{m!}\left(\frac{x}{1 - y}\right)^m\frac{1}{p!}x^p$$

$$= (1 - y)^{-b}F_2\left[a + d - 1;\, b,\, e;\, a,\, e;\, \frac{x}{1 - y},\, z\right] \quad . \tag{2.1}$$

Now taking $e = d$ and using the relationships (1.2) and (1.3) we obtain the reduction formulae

$$H_A[a + d - 1,\, b,\, c;\, a,\, c;\, x,\, y,\, z]$$

$$= (1 - x - y)^{-b}(1 - z)^{-d}G_2\left[b,\, d,\, 1 - a,\, 1 - d;\, \frac{x}{1 - x - y},\, \frac{z}{1 - z}\right] \quad , \tag{2.2}$$

$$H_A[a + d - 1, b, c; a, c; x, y, z] = (1 - x - y)^{-b} H_2\left[1 - d, b, a + d - 1, d; a; \frac{x}{x + y - 1}, -z\right] \cdot \quad (2.3)$$

Srivastava (1967, p. 111 (8.4, 8.5); 1968, p. 66 (8.4, 8.5)) was the first to have attempted to find these two formulae. His method, which was different from ours, was based on the use of the Pochhammer's double loop contour integral representation for the function H_A. However, some inaccuracies crept in and Srivastava (1967, p. 111 (8.4, 8.5)) ended up with $x/(1 - x)$ instead of $x/(1 - x - y)$ in (2.2) and with $x/(x - 1)$ instead of $x/(x + y - 1)$ in (2.3), while in another paper (Srivastava 1968, p. 66 (8.4)) he ended up with $x/(1 - z)$ and $z/(1 - x)$ instead of $x/(1 - x - y)$ and $z/(1 - z)$ respectively in (2.2).

2.2. Formula (2.1) leads us to a number of other reduction formulae for H_A. For example, on replacing d by $d + 1 - a$ and using a result from Bailey (1935, p. 79 (3)), we get formula (Srivastava 1967, p. 112 (8.6))

$$H_A[d, b, c; a, c; x, y, z] = (1 - y)^{-b}(1 - z)^{-d} {}_2F_1\left[b, d; a; \frac{x}{(1 - y)(1 - z)}\right] \cdot \quad (2.4)$$

Formula (2.1) in conjunction with another result from Bailey (1935, p. 102 (Ex. 20(i))) gives us

$$H_A[a + d - 1, d, c; a, c; x, y, z] = (1 - y)^{-d} F_4\left[a + d - 1; d; a, d; \frac{x(1 - z)}{1 - y}, \frac{z(1 - x - y)}{1 - y}\right] \cdot \quad (2.5)$$

Various other reduction formulae can be deduced using appropriate results from Burchnall (1942), Erdélyi (1948), Olsson (1964, 1965), etc.

The authors are grateful to the referees for their valuable suggestions to improve the paper.

REFERENCES

Bailey, W.N. 1935. Generalized hypergeometric series. Cambridge University Press, London.

Burchnall, J.L. 1942. Differential equations associated with hypergeometric functions. Quart. J. Math. Oxford Ser. **13:** 90–106.

Erdélyi, A. 1948. Transformations of hypergeometric functions of two variables. Proc. Roy. Soc. Edinb. Sect. A **62:** 378–85.

Erdélyi, A. 1950. Hypergeometric functions of two variables. Acta Math. **83:** 131–64.

Exton, H. 1976. Multiple hypergeometric functions and their applications. Ellis Horwood, Chichester, U.K.

Olsson, P.O.M. 1964. Integration of the partial differential equations for the hypergeometric functions F_1 and F_D of two and more variables. J. Math. Phys. **5** (3): 420–30.

Olsson, P.O.M. 1965. Certain solutions of the partial differential equations associated with Appell's hypergeometric function F_2. Ark. Fys. **29** (20): 285–91.

Srivastava, H.M. 1964. Hypergeometric functions of three variables. Ganita **15** (2): 97–108.

Srivastava, H.M. 1967. Some integrals representing triple hypergeometric functions. Rend. Circ. Mat. Palermo **16** (1): 99–115.

Srivastava, H.M. 1968. Some integrals representing triple hypergeometric functions. Math. Japon. **13** (1): 55–69.

(Received 18 May 1983, revised 13 October 1984)

حول اختزالية الدالة الثلاثية فوق الهندسية H_A

م . أ . قريشي و ف . ى . خان و م . أ . باثان
قسم الرياضيات بجامعة أليجار الاسلامية ، أليجار ، أوتر براديش ، الهند

خلاصة

في هذا البحث ، أمكن الحصول على بعض الدساتير الاختزالية للدالة الثلاثية فوق الهندسية H_A .