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Toeplitz operators on vector-valued Hardy spaces

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ABSTRACT

Let H be a separable Hilbert space, and $L^p(T, H)$ be the space of Bochner p-integrable functions with values in H. The associated vector-valued Hardy space is denoted by $H^p(T, H)$. For $\phi \in L^\infty(T, H)$, we define a Toeplitz-like operator T_ϕ: $H^2(T, H) \to H^2(T, H)$. The basic properties of T_ϕ are studied in this paper.

INTRODUCTION

Let T be the unit circle, and H a separable Hilbert space. If m is the normalized Lebesgue measure on T, we define a function f: $T \to H$ to be measurable if for each fixed $y \in H$, the scalar-valued function $t \to \langle f(t), y \rangle$ is Lebesgue measurable. The space $L^2(T, H)$ is the collection of all equivalence classes of measurable functions f: $T \to H$, which are Bochner square integrable, with

$$\|f\|^2 = \int_0^{2\pi} \|f(t)\|_H^2 dm(t) < \infty \quad .$$

The space $L^2(T, H)$ is a Hilbert space with respect to the inner product

$$\langle f, g \rangle = \int_0^{2\pi} \langle f(t), g(t) \rangle \, dm(t) \quad .$$

Moreover, there exists a one to one correspndence between the elements f of $L^2(T, H)$ and the sequences $\{a_k\}_{k=-\infty}^{+\infty} (a_k \in H)$ with $\Sigma_k \|a_k\|_H^2 < \infty$, such that for the corresponding f and $\{a_k\}_{k=-\infty}^{+\infty}$ we have

$$f(t) = \sum_{-\infty}^{+\infty} e^{ikt} a_k \qquad \text{and} \qquad \|f\|^2 = \sum_{-\infty}^{+\infty} \|a_k\|_H^2 \quad .$$

From this it follows that

$$a_k = \int_0^{2\pi} e^{-ikt} f(t) \, dm(t) \quad (k = 0, \pm 1, \pm 2, \ldots) \quad ,$$

a_k are called the Fourier coefficients of f. An important subspace of $L^2(T, H)$, which we shall denote by $H^2(T, H)$ consists of those functions $f \in L^2(T, H)$ for which

$a_k = 0 (k < 0)$. It is well known that $H^2(T, H)$ is a closed subspace of $L^2(T, H)$, and it consists of those analytic functions $u(\lambda) = \Sigma_{k=0}^{\infty} \lambda^k a_k$ in the unit disk with values in H such that

$$\int_0^{2\pi} \|u(re^{it})\|_H^2 \, dm(t) < \infty \quad (0 \leqslant r < 1) \quad .$$

Further $L^2(T, H) = H^2(T, H) \oplus (H^2(T, H))^{\perp}$, where $(H^2(T, H))^{\perp}$ is the orthogonal complement of $H^2(T, H)$.

The space $L^2(T, H)$ may be considered as the space of functions f which are represented as $f(t) = \Sigma_{n=1}^{\infty} f_n(t) e_n$, where $f_n(t) \in L^2(T)$, and $\{e_1, e_2, \ldots\}$ is some fixed orthonormal basis of H; also we have $\|f\|^2 = \Sigma_{n=1}^{\infty} \|f_n\|_2^2$. The space $L^{\infty}(T, H)$ is the space of all essentially bounded functions on T with values in H. It is clear that $L^{\infty}(T, H) \subseteq L^2(T, H)$. The space $H^{\infty}(T, H)$ is the space of all functions in $L^{\infty}(T, H)$ such that $a_k = 0$ for $k < 0$. Good references for the above material are Beals (1971), Sz. Nagy & Foias (1970), and Helson (1964).

Let z be a fixed element in H, such that $\|z\| = 1$, and $\phi \in L^{\infty}(T, H)$. The Laurent operator L_{ϕ} is defined on $L^2(T, H)$ by $L_{\phi} f = \phi f$, where $(\phi f)(t) = \langle \phi(t), z \rangle f(t)$. The Toeplitz operator T_{ϕ} is defined on $H^2(T, H)$ by $T_{\phi}(f) = P(L_{\phi} f)$, where P is the orthogonal projection of $L^2(T, H)$ onto $H^2(T, H)$. It is easily checked that L_{ϕ} and T_{ϕ} are bounded operators on $L^2(T, H)$ and $H^2(T, H)$ respectively. Moreover, $\|T_{\phi}\| \leqslant \|\phi\|_{\infty}$.

Toeplitz operators defined on $H^2(T)$ have been the object of much study (Douglas 1972). Also we would like to remark that Douglas (1973) studied Toeplitz operators T_{ϕ} on $H^2(T, L^2(C^n))$, where $\phi \in L^{\infty}(T, L^2(C^n))$. In section one of this paper some of the algebraic properties of Toeplitz operators defined on $H^2(T, H)$ are studied. Moreover, it is shown that for $\phi \in H^{\infty}(T, H)$, the Teoplitz operator T_{ϕ} is invertible if and only if ϕ is invertible in $H^{\infty}(T, H)$. Also, it is shown that if the real part of $\langle \phi(t), z \rangle$ is greater than or equal to ε, for some $\varepsilon > 0$, then T_{ϕ} is invertible. Moreover, it is proved that if H is finite dimensional, then the spectrum $\sigma(T_{\phi})$ of the Toeplitz operator T_{ϕ} is connected. In section two of this paper it is shown that the operator $T_{\phi} T_{\psi} - T_{\phi\psi}$ is compact, where $\phi \in L^{\infty}(T, H)$, and $\psi \in C(T, H)$, where $C(T, H)$ is the space of continuous functions from T to H.

Throughout this paper, I denotes the identity operator on appropriate space.

1. ALGEBRAIC AND SPECTRAL PROPERTIES OF TOEPLITZ OPERATORS

In this section some algebraic properties of Toeplitz operators are proved. In particular it is shown that the adjoint T_{ϕ}^* of T_{ϕ} is $T_{\bar{\phi}}$, where $\bar{\phi}$ is defined by $\langle \bar{\phi}(t), z \rangle = \overline{\langle \phi(t), z \rangle}$, $t \in T$. Moreover some spectral properties of T_{ϕ} are proved.

Proposition 1.1. Let $\phi \in L^{\infty}(T, H)$, then the adjoint T_{ϕ}^* of the Toeplitz operator T_{ϕ} is equal to $T_{\bar{\phi}}$.

Proof. Let $f, g \in H^2(T, H)$, then $\langle T_{\phi} f, g \rangle = \langle P(\phi f), g \rangle = \langle \phi f, g \rangle$. From the definition of the inner product, we have

$$\langle \phi f, g\rangle = \int_0^{2\pi} \langle (\phi f)(t), g(t)\rangle \, dm(t)$$

$$= \int_0^{2\pi} \langle \phi(t), z\rangle \langle f(t), g(t)\rangle \, dm(t)$$

$$= \int_0^{2\pi} \langle f(t), \overline{\langle \phi(t), z\rangle} g(t)\rangle \, dm(t) \quad .$$

It follows from the definition of $\bar{\phi}$ that

$$\langle T_\phi f, g\rangle = \int_0^{2\pi} \langle f(t), \langle \bar{\phi}(t), z\rangle g(t)\rangle \, dm(t)$$

$$= \langle f, \bar{\phi} g\rangle = \langle f, P(\bar{\phi} g)\rangle = \langle f, T_{\bar{\phi}} g\rangle \quad .$$

Therefore, the adjoint T_ϕ^* of T_ϕ is $T_{\bar{\phi}}$ and this ends the proof of the proposition.

Corollary 1.1. Suppose that $\langle \phi(t), z\rangle$ is real, then $T_\phi^* = T_\phi$.

Before we state and prove the next proposition, a lemma and some technical notations are needed. The space $H^2(T)$ is the usual Hardy space which consists of all complex-valued functions in $L^2(T)$ such that

$$\int_0^{2\pi} f(t) e^{int} \, dm(t) = 0 \qquad \text{for } n > 0 \quad .$$

The space $H^\infty(T)$ is the space of all complex-valued functions in $L^\infty(T)$ such that the previous condition is satisfied.

Lemma 1.1. Let $\psi \in H^\infty(T, H)$ and $\phi \in H^2(T, H)$. Then the function $\psi\phi \in H^2(T, H)$.

Proof. For any $t \in T$, $(\psi\phi)(t) = \langle \psi(t), z\rangle \, \phi(t)$. From properties of the Bochner integral it can be shown that $\langle \psi(t), z\rangle \in H^\infty(T)$. All we have to show now is that $\langle \psi(t), z\rangle \, \phi(t) \in H^2(T, H)$. Let $h(t) = \langle \psi(t), z\rangle$. Now $\phi(t) = \Sigma_{n=1}^\infty f_n(t) e_n$, where $f_n(t) \in H^2(T)$, and $\{e_1, e_2, \ldots\}$ is a fixed orthonormal basis of H. Moreover, $\|\phi\|^2 = \Sigma_{n=1}^\infty \|f_n\|_2^2$. Therefore, $h(t)\phi(t) = \Sigma_{n=1}^\infty h(t).f_n(t).e_n$. Note that since $h \in H^\infty(T)$ and $f_n \in H^2(T)$ for $n = 1, 2, \ldots$, then $h.f_n \in H^2(T)$. Note also that $\|h.\phi\|^2 = \Sigma_{n=1}^\infty \|h.f_n\|_2^2 < \infty$. From this it follows that $h.\phi \in H^2(T, H)$, and this ends the proof of the lemma.

Proposition 1.2. Let $\phi \in L^\infty(T, H)$, $\psi \in H^\infty(T, H)$. Then $T_\phi T_\psi = T_{\phi\psi}$.

Proof. Note that $T_\phi T_\psi f = T_\phi(P(\psi . f))$. Since $\psi \in H^\infty(T, H)$, and $f \in H^2(T, H)$, then it follows by Lemma 1.1, that $\psi . f \in H^2(T, H)$, and hence $T_\phi T_\psi f = P(\phi(\psi . f)) = T_{\phi\psi} f$.

Therefore, $T_\phi T_\psi = T_{\phi\psi}$ and that end the proof of the proposition.

In what follows some spectral properties of Toeplitz operators on $H^2(T, H)$ will be discussed. In particular the spectrum $\sigma(T_\phi)$ of self adjoint Toeplitz operators will be characterized, and the invertibility of Toeplitz operators with symbol in $H^\infty(T, H)$ will be discussed. Before, we do that, a definition and a technical notation are needed.

Definition 1.1. If S is an operator defined on the Hilbert space H, then the numerical range $W(S)$ of S is the set $\{\langle Sf, f\rangle : f \in H, \|f\| = 1\}$. The function $e \in L^2(T, H)$ defined by $e(t) = z$ is in $H^\infty(T, H)$. Moreover $ef = f$ for all $f \in L^2(T, H)$.

Proposition 1.3. Let $\phi \in L^\infty(T, H)$ such that $\mathrm{Re}(\langle \phi(t), z\rangle) \geqslant \varepsilon$ almost everywhere for some $\varepsilon > 0$. Then the Toeplitz operator T_ϕ is invertible.

Proof. Let $f \in H^2(T, H)$, such that $\|f\| = 1$. Then,

$$\begin{aligned}\langle T_\phi f, f\rangle &= \langle \phi f, f\rangle = \int_0^{2\pi} \langle (\phi f)(t), f(t)\rangle \, dm(t) \\ &= \int_0^{2\pi} \langle \phi(t), z\rangle \langle f(t), f(t)\rangle \, dm(t) \\ &= \int_0^{2\pi} \mathrm{Re}(\langle \phi(t), z\rangle) \langle f(t), f(t)\rangle \, dm(t) \\ &\quad + i \int_0^{2\pi} \mathrm{Im}(\langle \phi(t), z\rangle) \langle f(t), f(t)\rangle \, dm(t) \quad .\end{aligned}$$

Clearly 0 is not an element in $\overline{W(T_\phi)}$, the closure of the numerical range of T_ϕ. $\overline{W(T_\phi)}$. Since $\sigma(T_\phi) \subseteq \overline{W(T_\phi)}$, it follows that $0 \notin \sigma(T_\phi)$, and hence the Toeplitz operator T_ϕ is invertible, and this ends the proof of the proposition.

Theorem 1.1. Suppose $\langle \phi(t), z\rangle$ is real. Then

$$\sigma(T_\phi) = [\text{ess inf} \langle \phi(t), z\rangle, \text{ess sup} \langle \phi(t), z\rangle] \quad .$$

Proof. First we show that $\sigma(T_\phi) \subseteq [\text{ess inf} \langle \phi(t), z\rangle, \text{ess sup} \langle \phi(t), z\rangle]$. Since $\langle z, \phi(t)\rangle$ is real, then it follows by Corollary 1.1 that $T_\phi^* = T_\phi$, and hence $\sigma(T_\phi)$ is real. Let λ be a real number such that $\lambda \notin [\text{ess inf} \langle \phi(t), z\rangle, \text{ess sup} \langle \phi(t), z\rangle]$. Then it follows that $|\langle \phi(t), z\rangle - \lambda| \geqslant \delta > 0$ for some $\delta > 0$ almost everywhere. Therefore $\langle \phi(t), z\rangle - \lambda \geqslant \delta$, or $\langle \phi(t), z\rangle - \lambda \leqslant -\delta$. Hence by Proposition 1.3, $T_\phi - \lambda I$ is invertible, from which it follows that $\lambda \notin \sigma(T_\phi)$; and hence $\sigma(T_\phi) \subseteq [\text{ess inf} \langle \phi(t), z\rangle, \text{ess sup} \langle \phi(t), z\rangle]$. To finish the proof of the theorem, we need to show that $[\text{ess inf} \langle \phi(t), z\rangle, \text{ess sup} \langle \phi(t), z\rangle] \subseteq \sigma(T_\phi)$. Suppose that $T_\phi - \lambda I$ is invertible for λ real. Then there exists $g \in H^2(T, H)$, such that $(T_\phi - \lambda I)g = e$, where $e(t) = z$. Thus there exists an h in $(H^2(T, H))^\perp$ such that $(\phi - \lambda e)g = e + h$. From this it follows that $((\phi - \lambda e)g)(t) = (e + h)(t)$. Hence, $\langle (\phi - \lambda e)(t), z\rangle g(t) = z + h(t)$. Therefore, $\langle (\phi - \lambda e)(t), z\rangle . \langle g(t), z\rangle = 1 + \langle h(t), z\rangle$. Since h is in $(H^2(T, H))^\perp$, then $\langle h(t), z\rangle \in \overline{H_0^2}$, where $\overline{H_0^2}$ is the space of complex conjugates of functions in $H^2(T)$ which vanish at the origin. But $\langle (\phi - \lambda e)(t), z\rangle$ is real, hence $\langle (\phi - \lambda e)(t), z\rangle . \overline{\langle g(t), z\rangle} = 1 + \overline{\langle h(t), z\rangle}$, which is an element of $H^2(T)$. Therefore

$$\langle (\phi - \lambda e)(t), z\rangle . |\langle g(t), z\rangle|^2 = (1 + \overline{\langle h(t), z\rangle}) . \langle g(t), z\rangle \quad ,$$

which is an element of $H^1(T)$. From this it follows by Corollary 6.6, p. 150 (Douglas 1972) that $\langle(\phi - \lambda e)(t), z\rangle . |\langle g(t), z\rangle|^2 = c$, where c is constant. Since a non-zero analytic function cannot vanish on a set of positive measure, it follows that $\langle(\phi - \lambda e)(t), z\rangle$ will have the same sign of c, and this will imply that $\lambda \notin [\text{ess inf}\,\langle\phi(t), z\rangle, \text{ess sup}\,\langle\phi(t), z\rangle]$. Hence, $[\text{ess inf}\,\langle\phi(t), z\rangle, \text{ess sup}\,\langle\phi(t), z\rangle] \subseteq \sigma(T_\phi)$, and this ends the proof of the theorem.

Definition 1.2. Let $\phi \in L^\infty(T, H)$, then ϕ is invertible in $L^\infty(T, H)$ if there exists $\psi \in L^\infty(T, H)$ such that $\phi . \psi = e$.

We have to remark that if ϕ_1, ϕ_2 are in $L^\infty(T, H)$, then $L_{\phi_1\phi_2} = L_{\phi_1}L_{\phi_2} = L_{\phi_2\phi_1}$, although it is not necessary that $\phi_1\phi_2$ equals $\phi_2\phi_1$.

Theorem 1.2. If ϕ is a function in $L^\infty(T, H)$ such that the Toeplitz operator T_ϕ is invertible, then ϕ is invertible in $L^\infty(T, H)$.

Proof. Since T_ϕ is invertible, then there exists an $\varepsilon > 0$ such that $\|T_\phi g\| \geqslant \varepsilon . \|g\|$ for all $g \in H^2(T, H)$, and this will imply that $\|\phi g\| \geqslant \varepsilon\|g\|$ for all $g \in H^2(T, H)$. Hence

$$\int_0^{2\pi} |\langle\phi(t), z\rangle|^2 \|g(t)\|^2 \, dm(t) \geqslant \varepsilon^2 \int_0^{2\pi} \|g(t)\|^2 \, dm(t) \quad .$$

In particular, this is true for $g(t) = h(t) \otimes z$, where $h \in H^2(T)$. Therefore,

$$\int_0^{2\pi} |\langle\phi(t), z\rangle|^2 |h(t)|^2 \, dm(t) \geqslant \varepsilon^2 \int_0^{2\pi} |h(t)|^2 \, dm(t) \quad .$$

Following the argument of Douglass (1973, Proposition 7.6), it follows that $\langle\phi(t), z\rangle$ is invertible in $L^\infty(T)$, that is, there exist $h \in L^\infty(T)$ such that $\langle\phi(t), z\rangle h(t) = 1$. Consequently, $\langle\phi(t), z\rangle . h(t) \otimes z = 1 \otimes z$, where $h(t) \otimes z = h(t)z$. Therefore, $\phi(h \otimes z) = e$, hence, ϕ is invertible in $L^\infty(T, H)$, and this ends the proof of the theorem.

Corollary 1.2. Let $\phi \in L^\infty(T, H)$, then $\sigma(\phi) \subseteq \sigma(T_\phi)$.

Proof. It follows from Theorem 1.2.

Corollary 1.3. Let $\phi \in H^\infty(T, H)$, then the Toeplitz operator T_ϕ is invertible if and only if ϕ is invertible in $H^\infty(T, H)$.

Proof. If ϕ is invertible in $H^\infty(T, H)$, then it can be easily checked that the inverse of T_ϕ is $T_{\phi^{-1}}$. Suppose that T_ϕ is invertible, then there exists $\psi \in H^2(T, H)$ such that $T_\phi(\psi) = P(\phi . \psi) = e$. Since $\phi \in H^\infty(T, H)$, then it follows that $\phi . \psi \in H^2(T, H)$, and hence $T_\phi \psi = \phi . \psi = e$. From Theorem 1.2, we have ϕ is invertible in $L^\infty(T, H)$. Since $\psi \in H^2(T, H)$ and $\psi \in L^\infty(T, H)$, it follows that $\psi \in H^\infty(T, H)$, and this ends the proof of the corollary.

Theorem 1.3. Let H be an n-dimensional Hilbert space. Then the spectrum of T_ϕ is connected.

Proof. Since H is finite dimensional, then every $f \in L^2(T, H)$ can be written as $f = f_1 \otimes f_2 \otimes \ldots \otimes f_n$, where $f_i \in L^2(T)$, for $i = 1, 2, \ldots, n$ and $\|f\| = (\Sigma_{i=1}^n \|f_i\|^2)^{1/2}$. From this it can be easily shown that T_ϕ is the tensor product of T_ϕ^i on the Hilbert space tensor product

$$H^2(T, H) = H^2(T, H_1) \otimes H^2(T, H_2) \otimes \ldots \otimes H^2(T, H_n) \quad ,$$

where T_ϕ^i: $H^2(T, H_i) \to H^2(T, H_i)$, H_i is one-dimensional for $i = 1, 2, \ldots, n$, and is defined by $T_\phi^i f_i = P(\phi . f_i)$, where $(\phi f_i)(t) = \langle \phi(t), z \rangle f_i(t)$. Note that $\phi . f_i \in L^2(T)$, for $i = 1, 2, \ldots, n$. Since $\sigma(T_\phi^i)$ is connected for $i = 1, 2, \ldots, n$ (Douglas 1972), then it follows by Schechter (1969) that $\sigma(T_\phi) = \{\lambda_1 \lambda_2 \ldots \lambda_n : \lambda_i \in \sigma(T_\phi^i)\}$. To show that $\sigma(T_\phi)$ is connected, we have to note that $\sigma(T_\phi^1) \times \sigma(T_\phi^2) \ldots \sigma(T_\phi^n)$ is connected in the product topology. Further, the function θ

$$\theta: \sigma(T_{\phi_1}) \times \sigma(T_{\phi_2}) \times \ldots \times \sigma(T_{\phi_n}) \to \mathbb{C}$$

defined by $\theta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 \lambda_2 \ldots \lambda_n$ is continuous, and this ends the proof of the proposition.

2. ON COMPACTNESS OF $T_\psi T_\phi - T_{\psi\phi}$

In this section, it is proved that if $\phi \in C(T, H)$, and $\psi \in L^\infty(T, H)$, then the operator $T_\psi T_\phi - T_{\psi\phi}$ is compact. Before we prove the theorem a notation and a sequence of lemmas are needed.

Throughout this section $\phi(-n) = \chi(-n) \otimes \theta, \theta \in H$, where $(\chi(-n) \otimes \theta)(t) = e^{-int}\theta$, n is a positive integer.

Lemma 2.1 The operator $T_\psi T_{\phi(-1)} - T_{\psi\phi(-1)}$ is a rank one operator.

Proof. Let $f \in H^2(T, H)$, then $f(t) = \Sigma_{k=0}^\infty a_k e^{ikt}$, $a_k \in H$. Now, $T_\psi T_{\phi(-1)} f = T_\psi(P(\phi(-1)(f))$. However

$$(L_{\phi(-1)} f)(t) = \langle \theta, z \rangle e^{-it} f(t)$$
$$= \langle \theta, z \rangle \left(a_0 e^{-it} + \sum_{k=1}^\infty a_k e^{i(k-1)t} \right) \quad .$$

Further $a_0 e^{-it}$ is in the orthogonal complement of $H^2(T, H)$, where

$$a_0 = \langle f, 1 \rangle = \int_0^{2\pi} f(t)\, dm(t) = \hat{f}(0) \quad .$$

This implies that

$$T_{\phi(-1)} f = \langle \theta, z \rangle (e^{-it} f - \hat{f}(0)\, e^{-it}).$$

Therefore

$$T_\psi T_{\phi(-1)} f = P(L_\psi(\langle \theta, z \rangle e^{-it} f) - L_\psi(\hat{f}(0) \langle \theta, z \rangle e^{-it}))$$
$$= P(L_{\psi\phi(-1)} f - \langle \theta, z \rangle \hat{f}(0) P(e^{-it}\psi))$$
$$= T_{\psi\phi(-1)} f - \langle \theta, z \rangle \hat{f}(0) P(e^{-it}\psi) \quad .$$

Hence $T_\psi T_{\phi(-1)} - T_{\psi\phi(-1)}$ is a rank one operator, and this ends the proof of the lemma.

Before we state the next lemma, we have to remark that if $\langle\theta, z\rangle = 0$, then $T_{\phi(-n)} T_{\phi(-1)} = T_{\phi(-n-1)}$ is equal to the zero operator, so we assume that $\langle\theta, z\rangle$ is different from zero.

Lemma 2.2. Let $\tilde{\phi}(-1) = (1/\langle\theta, z\rangle)\chi(-n) \otimes \theta$, n is a positive integer, then $T_{\phi(-n)} T_{\tilde{\phi}(-1)} = T_{\phi(-n-1)}$.

Proof.

$$\begin{aligned}
(T_{\phi(-n)} T_{\tilde{\phi}(-1)})f &= P(L_{\phi(-n)} P(L_{\tilde{\phi}(-1)} f)) \\
&= P\left(\langle\theta, z\rangle\, e^{-int} \sum_{k=1}^{\infty} a_k e^{i(k-1)t}\right) \\
&= P\left(\langle\theta, z\rangle \sum_{k=1}^{\infty} a_k e^{i(k-n-1)t}\right) \\
&= \langle\theta, z\rangle \sum_{k=n+1}^{\infty} e^{i(k-n-1)t} \\
&= T_{\phi(-n-1)} f \quad ,
\end{aligned}$$

and this ends the proof of the lemma.

Lemma 2.3. The operator $T_\psi T_{\phi(-n)} - T_{\psi\phi(-n)}$ is compact for every integer n.

Proof. We have to remark that if n is a negative integer, then $\phi(-n) \in H^\infty(T, H)$, and hence by Proposition 1.2, we have $T_\psi T_{\phi(-n)} - T_{\psi\phi(-n)}$ is equal to the zero operator. Suppose that $T_\psi T_{\psi\phi(-N)} - T_{\psi\phi(-N)}$ is compact for N a positive integer. To finish the proof we have to show that $T_\psi T_{\phi(-N-1)} - T_{\psi\phi(-N-1)}$ is compact. From Lemmas 2.1, and 2.2, we have

$$\begin{aligned}
T_\psi T_{\phi(-N-1)} - T_{\psi\phi(-N-1)} &= (T_\psi T_{\phi(-N)} - T_{\psi\phi(-N)})\, T_{\tilde{\phi}(-1)} \\
&\quad + (T_{\psi\phi(-N)} \cdot T_{\tilde{\phi}(-1)} - T_{\psi\phi(-N)\tilde{\phi}(-1)}) \quad .
\end{aligned}$$

Since $T_\psi T_{\phi(-N)} - T_{\psi\phi(-N)}$ is compact by assumption, and by Lemma 2.1, $T_{\psi\phi(-N)} \cdot T_{\tilde{\phi}(-1)} - T_{\psi\phi(-N)\tilde{\phi}(-1)}$ is compact. It follows that $T_\psi T_{\phi(-N-1)} - T_{\psi\phi(-N-1)}$ is compact for every integer n, and this ends the proof of the lemma.

Theorem 2.1. Let $\psi \in C(T, H)$. Then the operator $T_\psi T_\phi - T_{\psi\phi}$ is compact, where $\psi \in L^\infty(T, H)$.

Proof. It is well known that $C(T, H)$ is isometrically isomorphic to the injective tensor product $C(T) \check{\otimes} H$ (see Diestel & Uhl 1977). Hence, elements of the form $P_k \otimes \theta$ span a dense subspace of $C(T) \check{\otimes} H$, where $\theta \in H$, and P_k is a trignometric polynomial of degree $\leqslant k$ in $C(T)$. So

$$C(T, H) = \overline{\text{span}\,\{P_k \otimes \theta\colon k \text{ is an integer}, \theta \in H\}} \quad .$$

But

$$P_k \otimes \theta = \sum_{-k}^{k} \alpha_m \chi(m) \otimes \theta \quad \text{, where } (P_k \otimes \theta)(t) = \sum_{-k}^{k} \alpha_m e^{imt}.\theta \quad ,$$

and α_m are complex numbers. Thus it follows from Lemma 2.3 that the operator $T_\psi T_{P_k \otimes \theta} - T_{\psi P_k \otimes \theta}$ is compact for all intergers k. Now let $\psi \in C(T) \otimes H$, then there exists a sequence $\{\phi_n\}_{n=1}^{\infty}$ in span $\{P_k \otimes \theta$: k is an integer, $\theta \in H\}$ such that $\lim_{n\to\infty} \phi_n = \phi$. Since $\psi \in L^\infty(T, H)$, then it follows that $\lim_{n\to\infty} \psi\phi_n = \psi\phi$. Now, since $\| T_{\psi\phi n} - T_{\psi\phi} \| \leqslant \| \psi\phi_n - \psi\phi \|$, it follows that $\lim_{n\to\infty} T_{\psi\phi_n} = T_{\psi\phi}$. From this it can be concluded that $\lim_{n\to\infty} (T_\psi T_{\phi_n} - T_{\psi\phi_n}) = T_\psi T_\phi - T_{\psi\phi}$. But the operator $T_\psi T_{\phi_n} - T_{\psi\phi_n}$ is compact for every n, then it follows that $T_\psi T_\phi - T_{\psi\phi}$ is compact, because it is the uniform limit of compact operators, and this ends the proof of the proposition.

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مؤثرات توپلتز على فضاءات هاردي ذات القيم المتجهة

رشدي خليل و نزيه فاعور
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خلاصة

لنفرض أن H هو فضاء هلبرت انفصالي وأن $L^p(T, H)$ هو فضاء بوخنـر للدوال المتكاملة للرتبة p ذات القيم في H . ان فضاء هاردي المرافق ذا القيم المتجهة هو $H^p(T, H)$.

اذا كان $\phi \in L^\infty(T, H)$ ، فاننا نعرف شبيه مؤثر توپلتز $T_\phi : H^2(T, H) \to H^2(T, H)$. ان الخصائص الاساسية لهذا المؤثر T_ϕ قد درست في هذا البحث .