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Modified stress design criterion for bevel gears having a short back support

E. M. AL-SHAREEDAH

Department of Mechanical Engineering, University of Kuwait, P.O. Box 5969, Kuwait

ABSTRACT

Strength of bevel gears can be increased substantially if a web, linking together the back of their teeth, is provided. This back support can be obtained by either cutting gears through standard gear-manufacturing operations, or through the new technique of gear forging. Previous analysis had shown that a bevel gear with a full back support, extending from the root to the top surface of the gear teeth, possesses a geometric limiting characteristic if a pair of such gears were put to run together in a system. To overcome this limiting characteristic, a new bevel gear type with a short back support is proposed and analyzed. A new design procedure is developed and compared with the results of a three-dimensional finite element model and with the well known Lewis criterion.

NOMENCLATURE

a = dimension of a plate along the x-axis.
b = dimension of a plate along the y-axis.
c = distance from neutral centerline of cantilever.
d = pitch diameter.
h = thickness of a plate.
h_1, h_2, h_3, h_4 = dimensions defined in Fig. 6.
l = length of a cantilever beam.
p = circular pitch.
$w(x, y)$ = deflection function of a plate.
$u(x)$ = deflection function of a beam.
B = width of a cantilever beam.
B_1, B_2 = dimensions defined in Fig. 6.
E = Young's modulus of elasticity.
F = face length of teeth defined in Fig. 6.
I = area moment of inertia.
H = depth of a cantilever beam.
M = bending moment.
N_g = number of teeth in a gear.
N_p = number of teeth in a pinion.
P_0 = concentrated load.

P_t = tangential force component, Fig. 3.
ν = Poisson's ratio.
ξ = dimension location along the x-axis for a point.
η = dimension location along the y-axis for a point.
σ = maximum stress.

INTRODUCTION

Bevel gears are the most commonly used gears in industry and the most efficient devices in transmitting rotation between angularly disposed shafts. Power requirements in machines involving bevel gears, may amount to thousands of horse-power and in many applications they have been successfully operated at speeds exceeding 20,000 rpm.

Straight bevels are the oldest, simplest and most widely used. Most recent developments, regarding straight bevel gear, center around materials and processes used in the manufacturing of these gears (Spotts 1971). Also, active research and manufacturing developments have been done in the area of gear teeth finish, where new techniques and processes were employed to improve the strength, durability, noise control, quality and accuracy of gears (Dudley 1962).

The most recent development, in the area of manufacturing processes of straight bevel gears, is the forging process which is a relatively new process, different from that used to cut or machine gears, or from that of the powder metal gear technique. In this process, a precise billet is cut from bar steel stock, and is then heated in the forging temperature of 2250–2300°F (ASM Handbook 1961). At that temperature range and with the pressure of two forging dies, the material can flow from within the dies leaving a back shoulder as a result of this forging operation. The forging producing a back support (web) creates an innovated gear teeth configuration different from any gear teeth configuration obtained by conventional processes. The back support resulting from this forging operation can add considerable strength and durability to the teeth of the gear. Figs 1 and 2 show the difference between straight gear teeth with no back support and some types of forged (or cut) gears with supports attached to the back of the teeth, respectively.

Analysis of two types of bevel gears having a linking support between teeth (Fig. 2a and b) has been performed. The geometrical shape of the first type (Fig. 2a) has a limiting characteristic, since variations of parameters such as root, pitch, face angles, pitch diameter, middle distance, etc. change the shape of the tooth considerably and consequently, make impossible the generalization of this shape to different sizes of

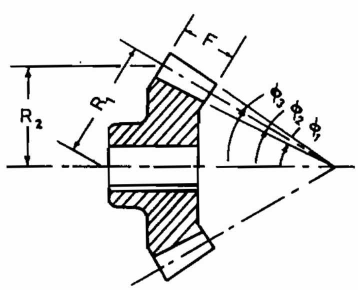

Fig. 1. Standard straight teeth bevel gears.

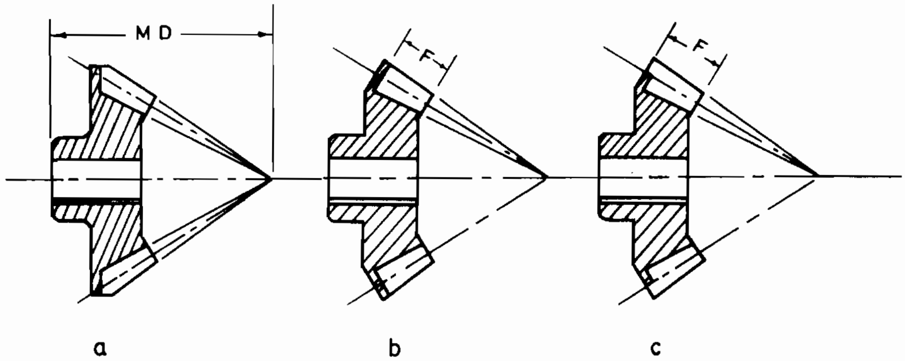

Fig. 2. Straight teeth bevel gears (a) upright back support (b) back support perpendicular to the line of the pitch cone (c) short back support.

bevel gears (Al-Shareedah & Lehnhoff 1984a). The second gear type (Fig. 2b) showed a geometric limiting characteristic also. In this case one pair of gears, having a full back support each, would result in a geometric conflict and interference if they were to mesh together (Al-Shareedah & Lehnhoff 1985). To avoid this geometric conflict, only one gear of the pair can have a full back support.

The purpose of investigating these different gears with back supports is to find out an optimal gear-support configuration that can add considerable strength and durability, while preserving the versatility of its usage (Al-Shareedah & Lehnhoff 1984a, 1984b, 1985; Al-Shareedah 1985). Therefore, an alternative geometric shape of teeth of a bevel gear of the types previously analyzed is proposed for investigation as shown in Fig. 2c.

REVIEW OF THE LEWIS CRITERION

When two bevel gears mesh together as a pair in a system, the load-carrying teeth rotate through a loading zone. The number of pairs of teeth in contact simultaneously varies from one to two, or even more. It is customary to use the Lewis design criterion to determine the ability of a single tooth to sustain a bending load when the load is acting at the corner or at the most unfavorable point, as indicated in Fig. 3 (Lewis 1893). The load along the pressure line in Fig. 3 is considered as being applied at the tip of the tooth, where it is resolved into radial and tangential components. The radial component causes a uniform compressive stress over the cross section, and it is customary to neglect this force when making stress calculations by this criterion. The tangential component produces a bending moment at the base which is the weakest section of the tooth, and therefore the maximum stress concentration occurs at that section.

To calculate the bending stress, Lewis assumed that the tooth is a cantilever beam and he used the elementary beam-bending equation to calculate stresses

$$\sigma = Mc/I = 6M/BH^2 \tag{1}$$

If the value of the moment M, as taken from Fig. 3, is substituted into Equation 1,

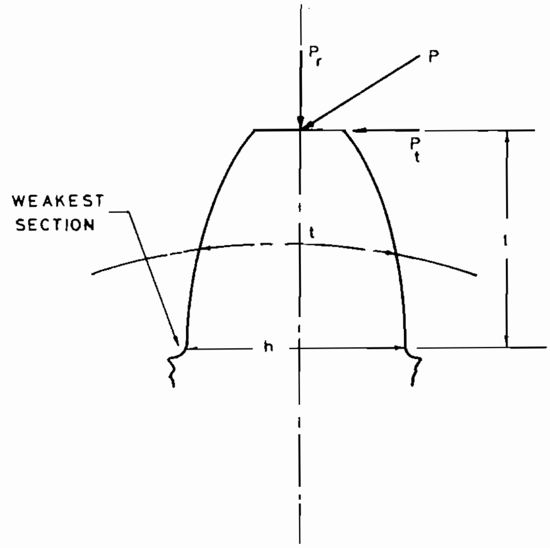

Fig. 3. Beam strength of a gear tooth.

the following Lewis equation results:

$$\sigma = \frac{P_t}{B}\frac{6l}{H^2} , \tag{2}$$

where the factor $H^2/6l$ is a purely geometrical property of the size and shape of the tooth and may be written as a function of the circular pitch. The ratio

$$y = \frac{H^2}{6lp} , \tag{3}$$

where

$$p = \frac{\pi d}{Ng} \tag{4}$$

is dimensionless and is called the Lewis factor. It depends on the number of teeth in the gear and on the gearing system used. The dimensions H and l in Equation 3 must be used for the cross section, which makes $H^2/6l$ a minimum (Lewis 1893). Equation 2 can also be written in the form

$$\sigma = \frac{6P_t}{H^2} l/B , \tag{5}$$

where l/B is a dimensionless quantity.

Stresses obtained through this criterion, in which the gear tooth is assumed to behave as a long, uniform and thin beam, are approximate, since the tooth is short, thick and nonuniform in cross section. In addition, the equation is valid only for points at considerable distances away from the point of application of the load. Nevertheless, it is customary to use the Lewis criterion in designing gears and gear systems.

PLATE FORMULAS FOR TEETH WITH FULL BACK

Attempts have been made to analyze bevel gear tooth with the same shape configuration as shown in Fig. 2a, and to create a criterion analogous to that assumed by Lewis for this type of gear (Al-Shareedah & Lehnhoff 1984a). In this case of a tooth having supports at the base and on the back (Fig. 2a), a plate simulation of the gear tooth, similar to the cantilever beam analogy used by Lewis, was considered. The plate was assumed to have two of its adjacent edges rigidly supported as shown in Fig. 4.

In solving the problem of Fig. 2a, a deflection function of a plate was assumed to be in the form of a polynomial function of the form

$$w(x, y) = a_1 + a_2x + a_3x^2 + a_4y + a_5xy + a_6x^2y + a_7y^2 + a_8xy^2 + a_9x^2y^2 \quad . \tag{6}$$

Considering the boundary conditions, this equation reduces to

$$w(x, y) = Ax^2y^2 \quad . \tag{7}$$

By using the Galerkin variational approach (Forray 1968), the constant A of Equation 7 is then calculated and is shown to be of the form

$$A = \frac{P}{D(Q_1 + Q_2 + Q_3)} \quad , \tag{8}$$

where

$$Q_1 = 4a\left(\frac{b^5}{5} + va^2\frac{b^3}{3}\right) \quad , \tag{9}$$

$$Q_2 = 4b\left(\frac{a^5}{5} + vb^2\frac{a^3}{3}\right) \quad , \tag{10}$$

$$Q_3 = \frac{16}{3}a^3b^3\left(-\tfrac{1}{3} + v\right) \quad , \tag{11}$$

and

$$D = \frac{Eh^3}{12(1 - v^2)} \quad . \tag{12}$$

The loading function P is specified by the series

$$P = \sum_{m=1}^{M}\sum_{n=1}^{N} P_{mn}\left[\left(\frac{\pi^2m^2 - 2}{m^3\pi^3}\right)a^3(-1)^m + \frac{2a^3}{m^3\pi^3}\right] \times \left[\left(\frac{\pi^2n^2 - 2}{n^3\pi^3}\right)b^3(-1)^n + \frac{2b^3}{n^3\pi^3}\right] \quad , \tag{13}$$

where

$$P_{mn} = \frac{4P_0}{ab}\sin\frac{m\pi}{a}\xi\sin\frac{n\pi}{b}\eta \quad . \tag{14}$$

In this case, the load is assumed to be a concentrated one acting on the plate (Fig. 4).

The stress design equation obtained from the analysis of the bending of a plate rigidly supported at two adjacent edges and under the action of a concentrated load

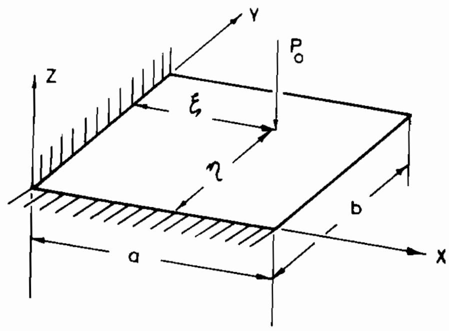

Fig. 4. A plate supported at two adjacent edges with a concentrated load.

is given by the expression

$$\sigma = \frac{48 b^4 a^4 P_0}{(h_1 + h_2)^2 (Q_1 + Q_2 + Q_3)} \sum_{m=1}^{M} \sum_{n=1}^{N} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}$$

$$\times \left[\left(\frac{\pi m^2 - 2}{m^3 \pi^3} \right) a^3 (-1)^m + \frac{2a^3}{m^3 \pi^3} \right] \left[\left(\frac{\pi m^2 - 2}{n^3 \pi^3} \right) b^3 (-1)^n + \frac{2b^3}{n^3 \pi^3} \right], \quad (15)$$

where h_1 and h_2 are the base and top thicknesses of the smallest section of the gear tooth, respectively.

Another plate formula was also investigated for the case of the shape configuration of Fig. 2b, where the back support is perpendicular to the line of the pitch cone. In this situation, the virtual plate was also assumed to be a rectangular plate with two of its adjacent edges rigidly supported. The deflection function is then assumed to be trigonometric of the form

$$w(x, y) = C \left(1 - \cos \frac{\pi x}{2a} \right) \left(1 - \cos \frac{\pi y}{2b} \right) . \quad (16)$$

It should be noted here that this deflection function is also a polynomial expression similar to Equation 6 if the trigonometric functions were to be expanded.

By using the Ritz variational approach (Forray 1968), constant C in Equation 16 is calculated and found to be of the form

$$C = \frac{2Q}{Df} \quad (17)$$

where

$$f = 3\left(\frac{\pi}{2a}\right)^4 \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) - 2\left(\frac{a}{2}\right)\left(\frac{\pi}{2a}\right)^4 \left(\frac{2b}{\pi}\right) + 2\nu\left(\frac{\pi}{2a}\right)\left(\frac{\pi}{2b}\right)$$

$$- 2\nu\left(\frac{a}{2}\right)\left(\frac{\pi}{2a}\right)^2\left(\frac{\pi}{2b}\right) - 2\nu\left(\frac{b}{2}\right)\left(\frac{\pi}{2a}\right)\left(\frac{\pi}{2b}\right)^2$$

$$+ 3\left(\frac{\pi}{2b}\right)^4\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) - 2\left(\frac{b}{2}\right)\left(\frac{\pi}{2b}\right)^4\left(\frac{2a}{\pi}\right)$$

$$+ \nu\left(\frac{\pi}{2a}\right)^2\left(\frac{\pi}{2b}\right)^2\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) \quad (18)$$

and Q is a function of the load of the form

$$Q = \int_0^a \int_0^b p(x, y) \left(1 - \cos \frac{\pi}{2a} x\right)\left(1 - \cos \frac{\pi y}{2b}\right) dx\, dy \quad . \tag{19}$$

For distributed loads, the function Q becomes

$$Q = GP \quad , \tag{20}$$

where the dimensionless quantity G is given by

$$G = 0{\cdot}132045 \quad . \tag{21}$$

The bending stress equation for this type of tooth is calculated using the theory of thin plates, where the maximum stress is reduced to an expression similar to the cantilever beam Equation 5 as indicated in Equation 22.

$$\sigma = (12P_0/h_p^2)GF \quad , \tag{22}$$

where the dimensionless geometric factor GF is given by the expression

$$GF = \left(\frac{\pi}{2b}\right)^2 \frac{G}{f} \quad . \tag{23}$$

In both cases of tooth shape shown in Figs 2a and 2b, the plate-based stress design equations were compared with finite element and experimental models. As a result of the existence of back supports, a considerable strength (as much as 50%) increase was obtained.

DESIGN CRITERION FOR BEVEL GEAR TEETH WITH SHORT BACK

The use of bevel gears having teeth with full back creates technical problems as mentioned before. These technical problems for making gears can be avoided and the teeth strength can be improved with respect to teeth without back, if the teeth with full back (Figs 2a and 2b) are replaced by teeth with short back (Fig. 2c).

In the following, an analysis of the case of teeth with short back is performed in order to define a design criterion of this type of bevel gear. The plate model, used for the case of full back, is not considered for teeth with short back, because of cumbersome expressions obtained [Equations 6, 15, 16 and 22] which require lengthy calculations which make the method unsuitable as a rapid tool of design for engineers and technicians. Instead, a simple criterion similar to that of Lewis will be derived and adopted.

If a plate were to be fixed in such a way that one side is rigidly supported along one of its sides, while an adjacent edge is supported rigidly only up to its middle point, and if the plate were also to be loaded in any fashion along the edge of the third side, a critical stress line joining the ends of the rigid supports will develop (Jones & Wood 1967; Park & Gumble 1971; Johansen 1972). This line represents the yield line of the plate, and therefore, the region of maximum stresses. This phenomenon is well predicted by the finite element model (Fig. 5) which represents schematically the surface elements of a typical bevel gear tooth with a maximum stress region indicated in black due to the application of a line load. The finite element model will be discussed in the next section.

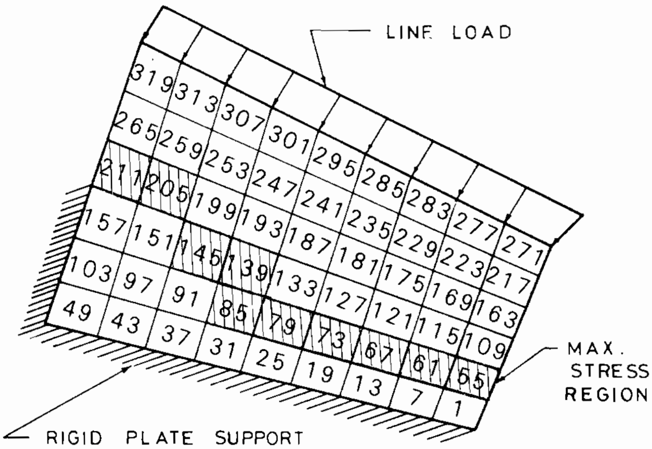

Fig. 5. Schematic representation of the surface elements of a bevel gear tooth with a line load and the maximum stress region.

For the purpose of design, a model using the fact indicated above is considered (Fig. 6). It shows the maximum stress section in the tooth of a bevel gear having a short back support. This cross section represents the region of maximum stress. To predict the stress in this critical cross section, a design criterion similar to that used by Lewis is adopted in the analysis.

For the bevel gear tooth shown in Fig. 6, it will be assumed that the tooth is supported rigidly at the maximum stress section, that is the section joining the ends of the rigid supports (Fig. 5), and that a fixed concentrated load is applied at the middle point of the free tip. In this case the stress can be approximated by using the cantilever beam criterion. To obtain the stress at the root of a cantilever beam, the following deflection function of the beam is used:

$$u(x) = \frac{P_0}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) , \tag{24}$$

where x is the coordinate along the axis directed from the cantilever rigid support

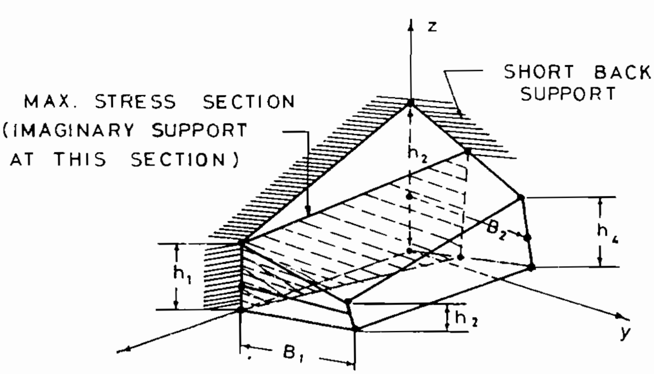

Fig. 6. Maximum stress section in a bevel gear tooth with short back support.

toward its free end. The stress is related to the deflection by

$$\sigma = Ec\frac{d^2u}{dx^2} \tag{25}$$

or

$$\sigma = \frac{cP_0}{I}(l - x) \ . \tag{26}$$

Therefore, the stress at the beam root is

$$\sigma = lc\frac{P_0}{I} \ . \tag{27}$$

This beam design equation depends on three parameters which should be determined using the tooth shape of Fig. 6. They are given by

$$l = \frac{2B_1 + B_2}{4} \ , \tag{28}$$

$$c = \frac{2h_1 + h_3 + h_4}{8} \ , \tag{29}$$

$$I = T\frac{(h_1^3 + k^3 + k^2h_1 + kh_1^2)}{48} \ , \tag{30}$$

where

$$T = F/\cos(\alpha) \ , \tag{31}$$

$$\alpha = \tan^{-1}(B_1/2F) \ , \tag{32}$$

and

$$k = \frac{h_3 + h_4}{2} \ . \tag{33}$$

If these parameters are substituted in Equation 27, then an expression similar to Equation 3, the Lewis formula, would result. This expression is

$$\sigma = 3P\frac{(2h_1 + h_3 + h_4)}{8(h_1^3 + k^3 + k^2h_1 + kh_1^2)}[(2B_1 + B_2)/T] \ , \tag{34}$$

where $(2B + B)/T$ is a dimensionless quantity.

Equation 34, represents the design equation of a bevel gear tooth with short back support. It can be considered as expressing a modified Lewis criterion. Equations 28 to 33 define the parameters of the cantilever tooth supported at the critical section (Fig. 6), involved in this modified criterion.

THE FINITE ELEMENT MODEL FOR BEVEL GEAR TEETH WITH SHORT BACK

In order to evaluate the modified criterion presented above, the bevel gear tooth shown in Fig. 2c is analyzed using the finite element technique. The results are compared to those of the modified and direct Lewis criterion. Due to the lack of symmetry of the tooth, a three-dimensional finite element model is required. Since it

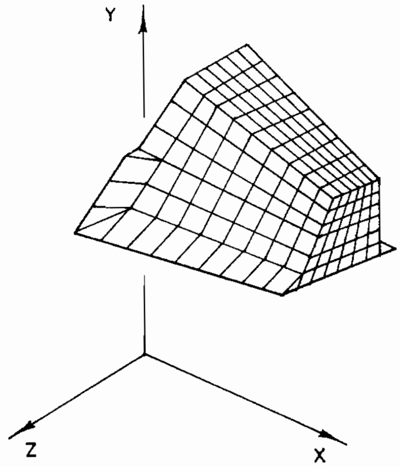

Fig. 7. Surface elements of a computer-generated finite element model.

was cumbersome to generate data for this problem manually, a mesh generation program was used. This program produces element data and coordinate data for each three-dimensional element for the entire gear tooth. The quadrilateral element was chosen to be the unit of division of the gear tooth in this program; this choice was dictated by considerations of the accuracy and generation simplicity of this type of element. The generated elements are arranged in layers extending from the tip to the back of the tooth (Fig. 7) and numbered in such a way that they create less sparsness of the matrix and, therefore, better accuracy.

The NASTRAN (NASA 1972) general purpose finite element program was used to analyze the stresses in each gear tooth. Fig. 7 represents the finite element model of the tooth showing the surface elements, while Fig. 8 represents a computer-generated three-dimensional plot of the gear tooth.

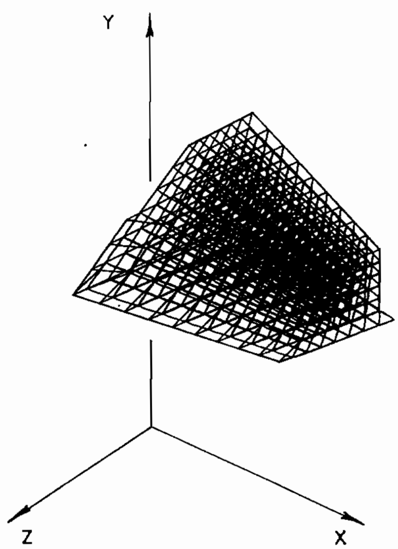

Fig. 8. Computer-generated finite element model of a bevel gear with a short back support.

The finite element model (Fig. 7) is the optimum model with respect to accuracy and number of elements. The optimized model is achieved by reaching a point at which no appreciable changes in stresses or deflections are apparent when the number of elements is increased. The particular flexibility of the element-generation program, where continuous subdivision and comparison of element factors are readily obtainable, made this optimized mesh easy to reach.

RESULTS AND DISCUSSION

In order to compare the modified cantilever model and the finite element model discussed above, to show their correlation with the well adopted Lewis criterion, a comprehensive finite-element stress analysis of gears of different diametral pitches and gear teeth ratios were performed using the NASTRAN general purpose finite element program. The results of typical finite element calculations, in addition to calculated results using the modified stress criterion, and the direct Lewis criterion are shown in Fig. 9. All these results are given for gears with short back supports, having a diametral pitch of 24, and teeth ratios ranging from 1·0 to 2·5 and teeth numbers ranging from 15 to 48.

The curves obtained show that the stresses predicted by the direct Lewis criterion have large differences in case of stresses obtained from the finite element model. The differences, which range between 31 and 66%, are to be expected because the Lewis criterion was not meant or established for such a case of a back support; while the

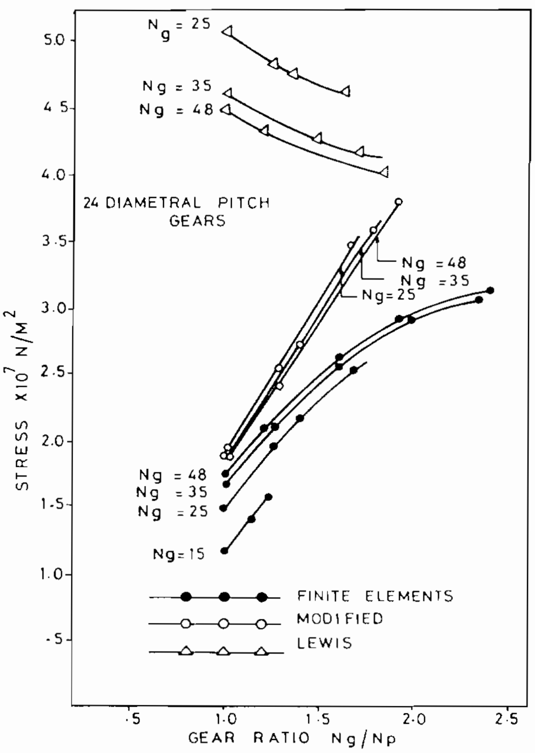

Fig. 9. Maximum stress of 24 diametral pitch gears for three different models.

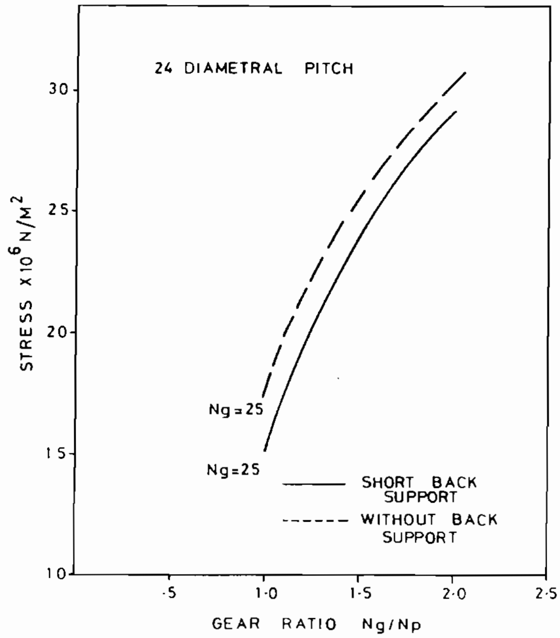

Fig. 10. Calculated stresses for 24 diametral pitch gears with short back supports.

difference is smaller when the finite element model is compared with the modified stress design criterion represented by Equation 34.

It is interesting to note (Fig. 9) that both the modified and the finite element stress curves have positive slopes, while the Lewis stress model has a negative slope when gear teeth ratios are increased. This negative slope of the Lewis curves is due to the nature of the factor y described in Equation 3, where y depends on the number of teeth in a gear and on the system of gearing used and calculated for cross sections where H and l make $H^2/6l$ a minimum length. In the analysis of Lewis criterion, quantitative rather than qualitative analysis was considered, in order to show the inapplicability of the Lewis criterion to this new type of gear.

It is also interesting to note that the finite element stress model, with short back support, has a difference of approximately 15% from the finite element model without back support as indicated in Fig. 10. This difference appears to be small when compared with the difference in stress obtained from the same models having a full back support. On the other hand, this difference is substantial when a pair of gears, having both short back supports, is compared with a pair of gears in which only one gear has a full back support (because only one full back support is permitted in that system to avoid an interference between the two gears).

CONCLUSIONS

As seen from the above analysis, the new modified design criterion for bevel gears having short back supports is more compatible with the finite element model than the Lewis design criterion.

It can also be concluded that the strength obtained from the new bevel gear configuration is substantial when a pair of gears is used as part of a gearing system. The increase in strength becomes clear when strength of pairs of gears with short back supports is compared with the strength of pairs of gears having full back supports, since in this latter case only one of the two gears meshing together is allowed to have a full back support.

The modified design criterion, also, adds further information to the sequence of investigations already performed, in order that an optimization of bevel gears with back supports can be achieved.

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طريقة معدلة لتصميم الجهد في
المسننات المخروطية التي تحتوي على اكتاف خلفية

ابراهيم محمد الشريدة
قسم الهندسة الميكانيكية بجامعة الكويت

خلاصة

أن قوة المسننات المخروطية يمكن زيادتها بدرجة كبيرة اذا ما اضيف لها حوامل تربط الأسنان من الخلف . هذه الحوامل يمكن الحصول عليها بواسطة طرق الضغط التشكيلي المستحدثة .

لقد اتضح من التحاليل السابقة أن المسننات المخروطية التي تحتوي على حوامل خلفية كاملة تشكل خواص هندسية محدودة ، اذا ما وضع زوج من هذه المسننات في جهاز واحد للعمل . وللتغلب على هذه الخاصية التي تحد من استعمال المسننات يقترح البحث مسننات مخروطية تحتوي على اكتاف قصيرة ، وتم تحليل هذه المسننات . كذلك يقدم البحث طريقة جديدة للتصميم وتمت مقارنتها مع كل من طريقة الأجزاء المتناهية ذات الابعاد الثلاثة وطريقة لويس المعروفة .