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On a generalization of a theorem of Markiewicz

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ABSTRACT

In this paper we obtain a result concerning the degree of approximation of a class of functions by using triangular matrix means of the conjugate series of a Fourier series, which generalizes a recent result of Markiewicz (1980).

1. INTRODUCTION

Let $f(t)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let its Fourier series be given by

$$f(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \quad . \tag{1.1}$$

Then its conjugate series is

$$-\sum_{k=1}^{\infty} (b_k \cos kt - a_k \sin kt) \quad . \tag{1.2}$$

We shall use the following notations:

$$\psi_x(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\} \quad ,$$

$$\Psi(t) = \int_0^t |\psi_x(u)| du \quad ,$$

$$\tilde{f}(x) = \frac{-2}{\pi} \int_{0+}^{\pi} \frac{\psi_x(t)}{2 \tan t/2} dt \quad .$$

Let $\lambda_{n,k}$ $(k = 0, 1, \ldots, n, n = 0, 1, \ldots)$ be a triangular matrix of real or complex numbers and let

$$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k} s_k \quad . \tag{1.3}$$

A series $\Sigma\, u_n$ with the sequence of partial sums $\{s_n\}$ is said to be summable $(\wedge)$ to s if the sequence $\{\sigma_n\}$ tends to a finite limit s as n tends to infinity. We denote by $\tilde{\sigma}_n(x)$ the $\wedge$-means of the series (1.2) at $t = x$, where $s_0 = 0$.

In what follows we suppose that $\{\lambda_{n,k}\}$ is non-negative with $\Sigma_{k=0}^{n} \lambda_{n,k} = 1, n = 0, 1, 2, \ldots$, then the necessary and sufficient condition for the regularity of $\wedge$-method is

$$\lim_{n\to\infty} \lambda_{n,k} = 0, \quad k = 0, 1, 2, \ldots .$$

For

$$\lambda_{n,k} = \begin{cases} \frac{p_{n-k}}{P_k} & k \leqslant n \\ 0 & k > n, \end{cases}$$

where $P_n = \Sigma_{v=0}^{n} p_v \neq 0$, σ_n defined by Equation 1.3, is the same as Nörlund mean (N, p_n) generated by the sequence of coefficients $\{p_n\}$. Suppose $f(x)$ is an arbitrary function defined in a closed interval I and let

$$\omega(\delta, f) = \operatorname*{Sup}_{x_1, x_2 \in I} |f(x_1) - f(x_2)|, \qquad |x_1 - x_2| \leqslant \delta .$$

This function $\omega(\delta, f)$ is called the modulus of continuity of $f(x)$. We write

$$\omega_1(\delta) = \omega_1[f, x](\delta) = \operatorname*{Sup}_{|h|<\delta} \left\{ \frac{1}{2h} \int_{-h}^{h} |f(x + u) - f(x)|\, du \right\} .$$

It is clear that if $f(x) \in C^*$ $[0, 2\pi]$, then $\omega_1(\delta) \leqslant \omega(\delta, f)$, where C^* denotes the class of continuous and periodic functions in $[0, 2\pi]$.

Concerning the approximation of $\tilde{f}(x)$ by the Nörlund means $\tilde{t}_n(x)$ of the series (1.2) Markiewicz (1980) proved the following theorem:

Theorem A. If at a fixed point x, $\omega_1(f, \delta) < \infty$ for $\delta \in (0, \pi]$ and if

$$\int_0^{\pi} \frac{|\psi_x(t)|}{t}\, dt < \infty \quad ,$$

then

$$|\tilde{f}(x) - \tilde{t}_n(x)| = 0\,(1) \left\{ \int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t}\, dt + \frac{1}{P_n} \sum_{k=0}^{n} \frac{P_k}{k + 1}\, \omega_1 \left(\frac{\pi}{k + 1} \right) \right\} ,$$

where $\{p_n\}$ is a non-increasing sequence of positive numbers.

The object of this note is to generalize Theorem A by proving the following result. The corresponding result for Fourier series was recently obtained by Mazhar (1983).

Theorem. If $\omega_1(\delta) < \infty$ for $\delta \in (0, \pi]$,

$$\int_0^{\pi} \frac{|\psi_x(t)|}{t}\, dt \text{ exists}$$

and $\{\lambda_{n,k}\}$ is non-decreasing with respect to k, then

$$|\tilde{\sigma}_n(x) - \tilde{f}(x)| = 0\,(1) \left\{ \sum_{k=0}^{n} \frac{\omega_1\left(\dfrac{\pi}{k+1}\right)}{(k+1)} \sum_{v=0}^{k} \lambda_{n,n-v} + \int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t}\, dt \right\} .$$

2. LEMMA

In order to prove this theorem we need the following lemma:

Lemma. (McFadden 1942). If $\{\lambda_{n,k}\}$ is non-negative and non-decreasing with respect to k, then for $0 \leqslant a \leqslant b \leqslant \infty$, $0 < t \leqslant \pi$ and for every n

$$\left| \sum_{k=a}^{b} \lambda_{n,n-k}\, e^{i(n-k)t} \right| \leqslant BA_{n,n-\tau} \quad ,$$

where B is an absolute constant, τ is the integral part of $1/t$ and

$$A_{n,n-\tau} = \sum_{v=n-\tau}^{n} \lambda_{n,v} \quad .$$

3. PROOF OF THE THEOREM

Using the formula in Zygmund (1959, p. 50)

$$\tilde{S}_n(x) = \frac{-2}{\pi} \int_0^{\pi} \psi_x(t) \frac{[\cos t/2 - \cos (n + \tfrac{1}{2})t]}{2 \sin t/2}\, dt \quad ,$$

we have

$$\begin{aligned}
\tilde{\sigma}_n(x) - \tilde{f}(x) &= \sum_{k=0}^{n} \lambda_{n,k} [\tilde{S}_k - \tilde{f}(x)] \\
&= \sum_{k=0}^{n} \lambda_{n,k} \left[\frac{-2}{\pi} \int_0^{\pi} \psi_x(t) \frac{(\cos t/2 - \cos (k + \tfrac{1}{2})t}{2 \sin t/2}\, dt \right. \\
&\quad \left. + \frac{2}{\pi} \int_0^{\pi} \frac{\psi_x(t)}{2 \tan t/2}\, dt \right] \\
&= \frac{-2}{\pi} \int_0^{\frac{\pi}{n+1}} \sum_{k=0}^{n} \lambda_{n,k} \psi_x(t) \tilde{D}_k(t)\, dt \\
&\quad + \frac{2}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \sum_{k=0}^{n} \lambda_{n,k} \psi_x(t) \frac{\cos (k + \tfrac{1}{2})t}{2 \sin t/2} \\
&\quad + \frac{2}{\pi} \int_0^{\frac{\pi}{n+1}} \sum_{k=0}^{n} \lambda_{n,k} \frac{\psi_x(t)}{2 \tan t/2}\, dt \\
&= I_1 + I_2 + I_3, \quad \text{say,}
\end{aligned}$$

where

$$\tilde{D}_k(t) = \sum_{v=1}^{k} \sin vt = \frac{\cos t/2 - \cos (k + \tfrac{1}{2})t}{2 \sin t/2} \quad .$$

Now since

$$\Psi(t) = \int_0^t |\psi_x(u)|\, du = \tfrac{1}{2}\int_0^t |f(x+u) - f(x-u) - f(x) + f(x)|\, du$$
$$\leqslant t\omega_1(t) \quad , \tag{3.1}$$

$$|I_1| \leqslant \frac{2}{\pi}\int_0^{\frac{\pi}{n+1}} |\psi_x(t)| \sum_{k=0}^{n} \lambda_{n,k}(k+1)\, dt$$
$$\leqslant C\,(n+1)\,\Psi\left(\frac{\pi}{n+1}\right)$$
$$\leqslant C\,\omega_1\left(\frac{\pi}{n+1}\right)$$
$$= 0\left\{\sum_{k=0}^{n} \frac{\omega_1\left(\frac{\pi}{k+1}\right)}{(k+1)} \sum_{v=0}^{k} \lambda_{n,n-v}\right\} , \tag{3.2}$$

where C is a constant not necessarily the same at each occurrence. Also, it is obvious that

$$|I_3| \leqslant \frac{2}{\pi}\int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{2\tan t/2}\, dt \quad .$$

Thus

$$I_3 = 0\left\{\int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t}\, dt\right\}. \tag{3.3}$$

Suppose $\gamma_n(t)$ is a linear function in $[k, k+1]$ such that $\gamma_n(k) = \lambda_{n,n-k}$, $k = 0, 1, 2, \ldots$. Writing

$$F_n(t) = \int_0^t \gamma_n(u)\, du, \quad t \geqslant 0 \quad ,$$

we have

$$F_n(k) = \sum_{v=0}^{k-1} \int_v^{v+1} \gamma_n(u)\, du$$
$$= \sum_{v=0}^{k-1} \frac{\gamma_n(v+1) + \gamma_n(v)}{2}$$
$$= \sum_{v=0}^{k-1} \frac{\gamma_{n,n-v-1} + \lambda_{n,n-v}}{2}$$
$$\leqslant \sum_{v=0}^{k} \lambda_{n,n-v} \leqslant 2F_n(k) \quad . \tag{3.4}$$

It is clear, in view of the above Lemma, that

$$\left|\sum_{k=0}^{n} \lambda_{n,n-k} \cos\left(n - k + \tfrac{1}{2}\right)t\right| \leqslant B \sum_{k=0}^{\tau} \lambda_{n,n-k} \leqslant 2BF_n(\tau) \leqslant 2BF_n\left(\frac{\pi}{t}\right) .$$

Hence

$$|I_2| \leqslant C \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} F_n\left(\frac{\pi}{t}\right) dt .$$

Integrating by parts, the above integral is

$$= \left[\Psi(t) \cdot \frac{1}{t} \cdot F_n\left(\frac{\pi}{t}\right)\right]_{\frac{\pi}{n+1}}^{\pi} + \int_{\frac{\pi}{n+1}}^{\pi} \frac{\Psi(t)}{t^2} F_n\left(\frac{\pi}{t}\right) dt$$

$$+ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\Psi(t)}{t} \cdot F_n'\left(\frac{\pi}{t}\right) \cdot \frac{\pi}{t^2} dt$$

$$= I_{21} + I_{22} + I_{23}, \quad \text{say} .$$

Now

$$I_{21} = \frac{\Psi(\pi)}{\pi} F_n(1) - \frac{n+1}{\pi} \Psi\left(\frac{\pi}{n+1}\right) \cdot F_n(n+1) .$$

Using 3.4

$$\frac{1}{\pi} \Psi(\pi) F_n(1) \leqslant \omega_1(\pi) \cdot \sum_{\nu=0}^{1} \lambda_{n,n-\nu} \leqslant 2\omega_1(\pi)\lambda_{n,n}$$

$$\leqslant 2 \sum_{k=0}^{n} \omega_1\left(\frac{\pi}{k+1}\right) \lambda_{n,n-k}$$

$$\leqslant 2 \sum_{k=0}^{n} \frac{\omega_1[\pi/(k+1)]}{(k+1)} \sum_{\nu=0}^{k} \lambda_{n,n-\nu} ,$$

since

$$\sum_{\nu=0}^{k} \lambda_{n,n-\nu} \geqslant \lambda_{n,n-k} \sum_{\nu=0}^{k} 1 = (k+1)\lambda_{n,n-k}.$$

Similarly

$$\frac{n+1}{\pi} \Psi\left(\frac{\pi}{n+1}\right) F_n(n+1) \leqslant \omega_1\left(\frac{\pi}{n+1}\right) \cdot \sum_{k=0}^{n} \lambda_{n,n-k}$$

$$\leqslant \omega_1\left(\frac{\pi}{n+1}\right) \sum_{k=0}^{n} \frac{1}{(k+1)} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}$$

$$\leqslant \sum_{k=0}^{n} \frac{\omega_1[\pi/(k+1)]}{(k+1)} \sum_{\nu=0}^{k} \lambda_{n,n-\nu} .$$

Thus

$$I_{21} = 0\left(\sum_{k=0}^{n} \frac{\omega_1[\pi/(k+1)]}{(k+1)} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}\right) . \qquad (3.5)$$

Also by virtue of 3.1

$$
\begin{aligned}
I_{22} &= \int_{\frac{\pi}{n+1}}^{\pi} \frac{\Psi(t)}{t^2} F_n\left(\frac{\pi}{t}\right) dt \\
&= \int_{1}^{n+1} \frac{\Psi\left(\frac{\pi}{u}\right)}{\pi^2/u^2} F_n(u) \cdot \frac{\pi}{u^2}\, du \\
&\leqslant C \int_{1}^{n+1} \frac{\pi}{u}\, \omega_1\left(\frac{\pi}{u}\right) F_n(u)\, du \\
&= C \sum_{v=1}^{n} \int_{v}^{v+1} \frac{\omega_1\left(\frac{\pi}{u}\right)}{u} F_n(u)\, du \\
&\leqslant C \sum_{m=0}^{n-1} \frac{\omega_1\left(\frac{\pi}{m+1}\right)}{m+1} \sum_{k=0}^{m+1} \lambda_{n,n-k} \\
&= 0(1) \sum_{k=0}^{n} \frac{\omega_1\left(\frac{\pi}{k+1}\right)}{(k+1)} \sum_{v=0}^{k} \lambda_{n,n-v} \quad ,
\end{aligned} \tag{3.6}
$$

since

$$
\lambda_{n,n-m-1} \leqslant \lambda_{n,n-m} \leqslant \sum_{k=0}^{m} \lambda_{n,n-k} \quad .
$$

Similarly

$$
\begin{aligned}
I_{23} &= \int_{\frac{\pi}{n+1}}^{\pi} \frac{\Psi(t)}{t} F_n'\left(\frac{\pi}{t}\right) \cdot \frac{\pi}{t^2}\, dt \\
&\leqslant \int_{1}^{n+1} \omega_1\left(\frac{\pi}{t}\right) F_n'(t)\, dt \\
&= \sum_{k=1}^{n} \int_{k}^{k+1} \omega_1\left(\frac{\pi}{t}\right) \gamma_n(t)\, dt \\
&\leqslant \sum_{k=1}^{n} \omega_1\left(\frac{\pi}{k}\right) \int_{k}^{k+1} \gamma_n(t)\, dt \\
&\leqslant \sum_{k=1}^{n} \omega_1\left(\frac{\pi}{k}\right) \lambda_{n,n-k} \\
&\leqslant \sum_{k=0}^{n} \frac{\omega_1\left(\frac{\pi}{k+1}\right)}{(k+1)} \sum_{v=0}^{k} \lambda_{n,n-v} \quad .
\end{aligned} \tag{3.7}
$$

From estimates 3.2–3.3 and 3.5–3.7 we conclude that

$$|\tilde{\sigma}_n - \tilde{f}(x)| = 0\left\{\sum_{k=0}^{n} \frac{\omega_1\left(\frac{\pi}{k+1}\right)}{k+1} \sum_{v=0}^{k} \lambda_{n,n-v}\right\} + 0\left\{\int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t}\, dt\right\}$$

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في هذا البحث حصلنا على نتيجة تتعلق بدرجة تقريب صف معين من الدوال وذلك باستخدام متوسطات المصفوفات المثلثية للمتسلسلة المرافقة لمتسلسلة فورييه . وهذه النتيجة هي تعميم لنظرية توصل اليها ماركيڤيتش عام ١٩٨٠ .