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Schur multipliers on the unit circle

ROSHDI KHALIL*

Department of Mathematics, University of Kuwait

ABSTRACT

Let T be the unit circle and m be the normalized Lebesgue measure on T. Set $L^2(T)\hat{\otimes}L^2(T)$ to denote the completed projective tensor product of $L^2(T,m)$ with itself. A function Φ defined on $T\times T$ is called a Schur multiplier of $L^2(T)\hat{\otimes}L^2(T)$ if $\Phi\cdot\psi \in L^2(T)\hat{\otimes}L^2(T)$ for all $\psi \in L^2(T)\hat{\otimes}L^2(T)$, where $(\Phi\cdot\psi)(x,y)=\Phi(x,y)\cdot\psi(x,y)$. It is the object of this paper to characterise the Schur multipliers of $L^2(T)\hat{\otimes}L^2(T)$.

INTRODUCTION

Let T be the unit circle and m be the Lebesgue measure on T. We let $L^2(T,m)$ denote the space of square-integrable functions with respect to the Lebesgue measure on T, and if $f\in L^2(T,m)$, then

$$\|f\|_2 = \Big(\int_T |f|^2 \mathrm{dm}\Big)^{\frac{1}{2}} \quad .$$

The space $L^2(T)\hat{\otimes}L^2(T)$ is the completion of the tensor product of $L^2(T,m)$ with itself with respect to the projective norm. As is well known (Schatten 1960), $L^2(T)\hat{\otimes}L^2(T)$ can be realised as a space of functions on $T\times T$ in such a way that if $\psi\in L^2(T)\hat{\otimes}L^2(T)$, then ψ has a representation

$$\psi(x,y) = \sum_{n=0}^{\infty} f_n(x)g_n(y); \quad \sum_{0}^{\infty} \|f_n\|_2\cdot\|g_n\|_2 < \infty \quad .$$

The norm of the function ψ in $L^2(T)\hat{\otimes}L^2(T)$ is

$$\|\psi\|_{\tau r} = \inf\left\{\sum_{0}^{\infty} \|f_n\|_2\cdot\|g_n\|_2\right\},$$

where the infimum is taken over all representations

$$\psi(x,y) = \sum_{n=0}^{\infty} f_n(x)g_n(y) \quad .$$

A function Φ defined on $T\times T$ will be called a (Schur) multiplier of $L^2(T)\hat{\otimes}L^2(T)$ if $\Phi\cdot\psi\in L^2(T)\hat{\otimes}L^2(T)$ for all $\psi\in L^2(T)\hat{\otimes}L^2(T)$, where $(\Phi\cdot\psi)(x,y)=\Phi(x,y)\cdot\psi(x,y)$.

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Clearly as an operator on $L^2(T)\hat{\otimes}L^2(T)$, ϕ is bounded and its norm will be denoted by $\|\Phi\|_m$.

The Schur multipliers of different spaces were considered by many authors (Schur 1911; Bennett 1977; Khalil 1980, 1981). Khalil (1980) characterised the multipliers of $L^2(I,v)\hat{\otimes}L^2(I,v)$ where I is the unit interval and v is any Borel measure. It should be remarked that the separability of the measure–space (I,v) was essential in the proof of Theorem 3.2 of Khalil (1980). It is the object of this paper to characterise the Schur multipliers of $L^2(T)\hat{\otimes}L^2(T)$, where in this case the measure–space (T,m) is not separable.

Throughout this paper, $L^\infty(T\times T)$ will denote the space of essentially bounded functions on $(T\times T,m\otimes m)$. The injective tensor product of $L^2(T,m)$ with itself will be denoted by $L^2(T)\check{\otimes}L^2(T)$ (Diestel & Uhl 1977), and $C(T\times T)$ is the space of continuous function on $T\times T$. Finally, $\ell^2(Z)$ is the space of square summable sequences on the integers Z.

MULTIPLIERS OF $L^2(T)\hat{\otimes}L^2(T)$

Let Φ be a function defined on $T\times T$, and $m(L^2(T)\hat{\otimes}L^2(T))$ be the space of multipliers of $L^2(T)\hat{\otimes}L^2(T)$. Then we prove

Lemma 1. If $\Phi\in m(L^2(T)\hat{\otimes}L^2(T))$, then $\Phi\in L^\infty(T\times T)$ and $\|\Phi\|_\infty \leqslant \|\Phi\|_m$.

Proof. Since the constant function $1\otimes 1\in L^2(T)\hat{\otimes}L^2(T)$, it follows that $\Phi\in L^2(T)\hat{\otimes}L^2(T)\subseteq L^2(T\times T)\subseteq L^1(T\times T)$. Let (k_n) be a summability kernel for $L^1(T\times T)$. Consider the sequence $\Phi_n=\Phi * k_n$. Clearly $\Phi_n\in C(T\times T)$. We claim that $\Phi_n\in m(L^2(T)\hat{\otimes}L^2(T))$ for all n. To prove that, let K be an element in $L^2(T)\otimes L^2(T)$ of finite rank. So $K(x,y)$ is the kernel of a finite rank operator on $L^2(T)$. Let $f\otimes g$ be any atom of unit norm in $L^2(T)\hat{\otimes}L^2(T)$. Since $L^2(T)\hat{\otimes}L^2(T)=(L^2(T)\check{\otimes}L^2(T))^*$, we consider

$$\begin{aligned} & |<\Phi_n\cdot f\otimes g, K>| \\ &= |\iint \Phi_n(x,y)\, f(x)\, g(y)\, K(x,y)\, dzdy| \\ &= |\iint f(x)\, g(y)\, K(x,y) \iint \Phi(x-s,y-t)\, k_n(s,t)\, dsdt\, dxdy| \end{aligned}$$

An application of Fubini's Theorem implies

$$\begin{aligned} |<\Phi_n\cdot f\otimes g, K>| &\leqslant \|k_n\|_l\cdot\|\Phi\|_m\cdot\|K\|\cdot\|f\otimes g\|_{\tau r} \\ &\leqslant \|\Phi\|_m\cdot\|f\otimes g\|_{\tau r}, \end{aligned}$$

where $\|K\|$ is the norm of K in $L^2(T)\check{\otimes}L^2(T)$. Hence $\Phi_n\in m(L^2(T)\hat{\otimes}L^2(T))$ and $\|\Phi_n\|_m\leqslant\|\Phi\|_m$ for all n.

Now, we show that $\|\Phi_n\|_\infty\leqslant\|\Phi_n\|_m\leqslant\|\Phi\|_m$. Since $\Phi_n\in C(T\times T)$, there is a point $(a,b)\in T\times T$ such that $\|\Phi_n\|_\infty=|\Phi_n(a,b)|$. Without loss of generality we let $(a,b)=(1,1)$. The continuity of Φ_n on the compact space $T\times T$ implies that for every $\epsilon>0$, there exists a closed neighbourhood of $(1,1)$, say $V(1,1)$, such that $|\Phi_n(s,t)-\Phi_n(1,1)|<\epsilon$ for all $(s,t)\in V(1,1)$. So let $\epsilon>0$ be given and $V(1,1)$ be the corresponding neighbourhood of $(1,1)$. Let $f\otimes g\in L^2(T)\hat{\otimes}L^2(T)$ be such that $\|f\otimes g\|_{\tau r}=1$ and the support of $f\otimes g$ is contained in $V(1,1)$.
Then

$$\begin{aligned} \|\Phi_n\|_\infty &= |\Phi_n(1,1)| \\ &= \|\Phi_n(1,1)\cdot f\otimes g\|_{\tau r} = \|\Phi_n(1,1)\, f\otimes g\|_2 \end{aligned}$$

$$\leqslant \|(\Phi_n - \Phi_n(1,1)) \cdot f \otimes g\| + \| \Phi_n \cdot f \otimes g\|_{tr}$$
$$\leqslant \sup_{(s,t)\in V} | \Phi_n(s,t) - \Phi_n(1,1)| + \|\Phi_n\|_m$$
$$\leqslant \epsilon + \|\Phi_n\|_m \quad .$$

Since ϵ was arbitrary, we get $\|\Phi_n\|_\infty \leqslant \|\Phi_n\|_m \leqslant \|\Phi\|_m$.

Finally, since the sequence $\Phi_n \to \Phi$ in $L^1(T\times T)$, there exists a subsequence (Φ_{n_r}) which converges to Φ pointwise $a \cdot e\, m\otimes m$. But (Φ_{n_r}) is a uniformly bounded sequence in $L^\infty(T\times T)$. Hence, by Alaoglu's Theorem, there exist $\psi \in L^\infty(T\times T)$ and a subsequence of (Φ_{n_r}) which converges to ψ in the w^*-topology. Consequently $\Phi = \psi$ and $\|\Phi\|_\infty \leqslant \|\Phi\|_m$. This completes the proof of the lemma.

The basic properties of the set $m(L^2(T)\hat{\otimes} L^2(T)$ can be summarised in the following.

Theorem 1. The space $m(L^2(T)\hat{\otimes} L^2(T))$ is a maximal commutative Banach algebra of operators on $L^2(T)\hat{\otimes} L^2(T)$.

Proof. That $m(L^2(T)\hat{\otimes} L^2(T))$ is a commutative algebra follows from the definition of a multiplier together with Lemma 1.

To prove the completeness of $m(L^2(T)\hat{\otimes} L^2(T))$ we let (Φ_n) be a Cauchy sequence in $m(L^2(T)\hat{\otimes} L^2(T))$. Since $L^2(T)\hat{\otimes} L^2(T)$ is a Banach space, it follows that $(\Phi_n \,.\, \psi)$ converges for every $\psi \in L^2(T)\hat{\otimes} L^2(T)$. Define an operator

$$A: L^2(T) \hat{\otimes} L^2(T) \to L^2(T)$$

such that $A(\psi) = \lim_n \Phi_n \cdot \psi$. Clearly A is bounded linear operator on $L^2(T)\hat{\otimes} L^2(T)$.

Define

$$\Phi(x,y) = \frac{A(\psi)\,(x,y)}{\psi(x,y)} \quad .$$

Since Φ_n $m(L^2(T)\hat{\otimes} L^2(T))$ it follows that

$$G \cdot \Phi_n(\psi) = \psi \cdot \Phi_n(G) \tag{1}$$

for all ψ and G in $L^2(T)\hat{\otimes} L^2(T)$. Hence

$$G \cdot A(\psi) = \psi \cdot A(G) \tag{2}$$

for all ψ and G in $L^2(T)\hat{\otimes} L^2(T)$. Further, since $\Phi_n(\psi) = \Phi_n \cdot \psi$, we have

$$[G \cdot \Phi_n(\psi)](x,y) = [\psi \cdot \Phi_n(G)](x,y) \tag{3}$$

for every $(x,y)\in T\times T$. Relations (2) and (3) imply that $[G \cdot A(\psi)](x,y) = [\psi \cdot A(G)](x,y)$ for all ψ and G in $L^2(T)\hat{\otimes} L^2(T)$ and for all $(x,y)\in T\times T$. Consequently Φ is independent of ψ. Furthermore, if $\psi \equiv 0$ on a set $E \subseteq T\times T$, then we can choose G such that $G(x,y) \neq 0$ for all $(x,y)\in E$. It follows that $A(\psi)(x,y) = 0$ for all $(x,y)\in E$. Hence

$$A(\psi)\,(x,y) = \Phi(x,y) \cdot \psi(x,y) \quad .$$

To prove the maximality of $m(L^2(T)\hat{\otimes} L^2(T))$, let B be a bounded operator on $L^2(T)\hat{\otimes} L^2(T)$ which commutes with $m(L^2(T)\hat{\otimes} L^2(T))$. Put $J = B(1\otimes 1)$. Then $B(\Phi \cdot 1\otimes 1) = \Phi(B(1\otimes 1)) = \Phi \cdot J$ for all $\Phi \in m(L^2(T)\hat{\otimes} L^2(T))$. Let $f\otimes g \in L^2(T)\hat{\otimes} L^2(T)$. Choose (u_n) and (v_n) in $C(T)$ such that $u_n \to f, v_n \to g$ both in $L^2(T)$-norm and pointwise. By the Lebesgue dominated convergence Theorem we get that $u_n\otimes v_n \to f\otimes g$ in the w^*-topology of $L^2(T)\otimes L^2(T)$. That is

$$\langle u_n \otimes v_n, K\rangle \to \langle f \otimes g, K\rangle$$

for every $K \in L^2(T)\check{\otimes} L^2(T)$. This implies that

$$\langle B(u_n \otimes v_n), K\rangle \to \langle B(f \otimes g), K\rangle,$$

and hence

$$\langle J \cdot u_n \otimes v_n, K\rangle \to \langle B(f \otimes g), K\rangle \quad .$$

Consequently, $\langle J \cdot f\otimes g, K\rangle = \langle B(f\otimes g), K\rangle$. It follows from the Hahn–Banach Theorem that $B(f\otimes g) = J \cdot f\otimes g$ for all $f\otimes g \in L^2(T)\hat{\otimes} L^2(T)$. Thus $B \in m(L^2(T)\hat{\otimes} L^2(T))$. This completes the proof of the theorem.

Now, let Z be the group of integers, ℓ^2 be the space of square summable sequences on Z. If $A(Z)$ is the space of Fourier coefficients of functions in $L^1(T)$, then the dual space of $A(Z)$ will be denoted by $P(Z)$. Since $\ell^2 \subseteq A(Z)$, it follows that $P(Z) \subseteq \ell^2$ and $P(Z)$ is isometrically isomorphic to $L^\infty(T)$ (Larsen 1971).

By a convoluter of a function space, we mean a Fourier multiplier, which is an operator on the space that commutes with all translation operators. An excellent account on convoluters can be found in Larsen (1971). It follows from Plancherel's Theorem that $m(L^2(T))$ is isometrically isomorphic to the space of convoluters of ℓ^2. Consequently, to characterise $m(L^2(T)\hat{\otimes} L^2(T))$, it is enough to characterise the convoluters of $\ell^2\hat{\otimes}\ell^2$. If $\mathrm{Con}(X)$ denotes the convoluters of the space X, then $\mathrm{Con}(L^2(G)) \cong P(\hat{G})$ for any locally compact abelian group G, where $\hat{G}$ is the dual group (Larsen 1971).

Now, we introduce the space $N(Z \times Z)$ to be the space of all functions σ on $Z \times Z$ such that there exists a sequence (σ_n) in $P(Z)\hat{\otimes} P(Z)$ with the following properties:

(i) $\sigma_n \xrightarrow{n} \sigma$ *pointwise*

(ii) $\|\sigma_n\| \leqslant \lambda$ for all n, and for some constant $\lambda > 0$. where $\|\sigma_n\|$ is the norm of σ_n as a convolution operator on $\ell^2\hat{\otimes}\ell^2$.

Theorem 2. Let σ be a function on $Z \times Z$. If $\sigma \in N(Z \times Z)$, then $\sigma \in \mathrm{Con}(\ell^2\hat{\otimes}\ell^2)$.

Proof. Let $\sigma_n \in P(Z)\hat{\otimes} P(Z)$ such that $\sigma_n(r,s) \to \sigma(r,s)$ for all $(r,s) \in Z \times Z$ and $\|\sigma_n\| \leqslant \lambda$.

Let $f\otimes g$ be an element of compact support in $\ell^2\hat{\otimes}\ell^2$ and ψ be the kernel of an operator of finite rank with compact support in $\ell^2\check{\otimes}\ell^2$. Since $\sigma_n \in P(Z)\hat{\otimes} P(Z)$, it follows that $\sigma_n \in \mathrm{Con}(\ell^2\hat{\otimes}\ell^2)$ and $\sigma_n(f\otimes g) \in \ell^2\hat{\otimes}\ell^2$. Consider the sequence of numbers

$$\begin{aligned} a_n &= \langle \sigma_n(f \otimes g), \psi\rangle \\ &= \iint_{Z\times Z} \sigma_n(f \otimes g)(i,j)\, \psi(i,j)\, dv(i)\, dv(j) \end{aligned}$$

where v is the counting measure on Z. Now,

$$\begin{aligned} &|\sigma_n(f \otimes g)(i,j)| \\ &\leqslant \|\sigma_n(f \otimes g)\|_{tr} \\ &\leqslant \|\sigma_n\| \cdot \|f \otimes g\|_{tr} \\ &\leqslant \lambda \cdot \|f \otimes g\|_{tr} \quad . \end{aligned}$$

Further, since ψ is of compact support, it follows that $\psi \in \ell^1(Z \times Z)$. Hence, an application of the Lebesgue dominated convergence Theorem gives

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \langle \sigma_n(f \otimes g), \psi\rangle \\ &= \langle \lim_{n\to\infty} \sigma_n(f \otimes g), \psi\rangle. \end{aligned}$$

Now, $\sigma_n(i,j)=\langle\sigma_n,\ \delta_i\otimes\delta_j\rangle$, where $\delta_m(s)=\begin{cases}0 \text{ if } m\neq s\\ 1 \text{ if } m=s\end{cases}$.

It follows that

$$\begin{aligned}|\sigma_n(i,j)| &\leqslant \|\sigma_n\|_{P(Z\times Z)}\cdot\|\delta_i\otimes\delta_j\|_{A(Z\times Z)}\\ &\leqslant \|\sigma_n\|_{P(Z\times Z)}\leqslant\|\sigma_n\|\leqslant\lambda,\end{aligned}$$

where $\|\sigma_n\|_{P(Z\times Z)}$ is the norm of σ_n as a linear functional on $A(Z\times Z)$. So, another application of the Lebesgue dominated convergence Theorem implies that

$$\lim_{n\to\infty}\sigma_n(f\otimes g)\,(i,j)=\sigma(f\otimes g)\,(i,j),$$

where $\sigma_n(f\otimes g)(i,j)=(\sigma_n * f\otimes g)\ (i,j)$ and $\sigma(f\otimes g)\ (i,j)=(\sigma * f\otimes g)\ (i,j)$. It follows that the sequence $a_n=\langle\sigma_n(f\otimes g),\psi\rangle$ is a convergent sequence.

Define an operator $S_\sigma:\ell^2\hat{\otimes}\ell^2\to\ell^2\hat{\otimes}\ell^2$, such that $\langle S_\sigma(f\otimes g),\psi\rangle=\lim_n\langle\sigma_n(f\otimes g),\psi\rangle$. Clearly S_σ is linear, and

$$\begin{aligned}|<S_\sigma(f\otimes g),\psi>| &= \lim_n|<\sigma_n(f\otimes g),\psi>|\\ &\leqslant\lim_n\|\sigma_n\|\cdot\|f\otimes g\|_{tr}\cdot\|\psi\|\\ &\leqslant\lambda\|f\otimes g\|_{tr}\cdot\|\psi\|,\end{aligned}$$

where $\|\psi\|$ is the injective norm of ψ as an element in $\ell^2\check{\otimes}\ell^2$. Hence S_σ is bounded and $\|S_\sigma\|\leqslant\lambda$.

Further, if $\tau(x,y)$ is the translation operator on $\ell^2\hat{\otimes}\ell^2$, where $\tau(x,y)\ (f\otimes g)\ (i,j)=f(i-x)g(j-y)$, $x,y\in Z$, then,

$$\begin{aligned}&<\sigma\tau(x,y)\ (f\otimes g),\ \psi\rangle\\ &=\lim_n\langle\sigma_n\tau(x,y)\ (f\otimes g),\psi\rangle\\ &=\lim_n\ \langle\tau(x,y)\sigma_n(f\otimes g),\psi\rangle\\ &=\langle\tau(x,y)\ \sigma(f\otimes g),\ \psi\rangle\quad.\end{aligned}$$

Hence σ commutes with every translation operator. This implies that $\sigma\in\mathrm{Con}(\ell^2\hat{\otimes}\ell^2)$. This completes the proof of the theorem.

Theorem 3. Let $\sigma\in\mathrm{Con}(\ell^2\hat{\otimes}\ell^2)$. Then $\sigma\in N(Z\times Z)$.

Proof. Let (k_n) be a summability kernel in $L^1(T\times T)$ such that $\|k_n\|_1\leqslant 1$ and $\hat{k}_n$ is of compact support on $Z\times Z$ for all $n=1,2\ldots$, where $\hat{k}_n$ is the Fourier transform of k_n. Set $\sigma_n=\sigma\cdot\hat{k}_n$. Since σ_n has compact support, $\sigma_n\in P(Z)\hat{\otimes}P(Z)$. Further, since $\hat{k}_n(i,j)\to 1$ for all $(i,j)\in Z\times Z$, it follows that $\sigma_n(i,j)\to\sigma(i,j)$. Using the same argument as in Theorem 2, we get

$$\sigma_n(f\otimes g)\ (i,j)\to\sigma(f\otimes g)\ (i,j)$$

for all $(i,j)\in Z\times Z$, and for all $f\otimes g\in\ell^2\hat{\otimes}\ell^2$. Further

$$\begin{aligned}|\sigma_n(f\otimes g)\ (i,j)| &= |\langle\sigma_n *(f\otimes g),\ \delta_i\otimes\delta_j\rangle|\\ &\leqslant|\langle\sigma,\ \hat{k}_n\,(\tilde{f}*\delta_i)\otimes(\tilde{g}*\delta_j)\rangle|\\ &\leqslant\|\sigma\|_{P(Z\times Z)}\cdot\|k_n\|_1\cdot\|f\otimes g\|_2\\ &\leqslant\|\sigma\|_{P(Z\times Z)}\cdot\|f\otimes g\|_2,\end{aligned}$$

where $\tilde{f}(x)=f(-x)$. Hence, using the Lebesgue dominated convergence Theorem we get

$$\langle \sigma_n(f \otimes g), \psi \rangle \rightarrow \langle \sigma(f \otimes g\rangle, \psi \rangle$$

for $f \otimes g \in \ell^2 \hat{\otimes} \ell^2$ and $\psi \in \ell^2 \check{\otimes} \ell^2$. Thus the sequence of operators (σ_n) converges to σ in the weak-operator topology. It follows (Maddox 1970) that (σ_n) is a uniformly bounded sequence of operators on $\ell^2 \hat{\otimes} \ell^2$. This completes the proof of the theorem.

Theorem 2 and Theorem 3 now give the main result.

Theorem 4. Let $\Phi \in L^\infty(T \times T)$. Then the following are equivalent:

(i) $\Phi \in m(L^2(T) \hat{\otimes} L^2(T))$
(ii) $\hat{\Phi} \in N(Z \times Z)$.

As an application of Theorem 4:

Clearly $L^\infty(T) \hat{\otimes} L^\infty(T) \subseteq m(L^2(T) \hat{\otimes} L^2(T))$. Let $L^\infty(T) \hat{\hat{\otimes}} L^\infty(T)$ be the space of all $\Phi \in L^\infty(T \times T)$ such that there exist $\Phi_n \in L^\infty(T) \hat{\otimes} L^\infty(T)$ with $\|\Phi_n\|_m < \lambda$ for all n, and $\langle \Phi_n, f \rangle \rightarrow \langle \Phi, f \rangle$ for all $f \in L^1(T \times T)$. It follows from Theorem 4 that $m(L^2(T) \hat{\otimes} L^2(T) = L^\infty(T) \hat{\hat{\otimes}} L^\infty(T)$.

REFERENCES

Bennett, G. 1977. Schur multipliers. Duke J. Math. **44:** 603–39.

Diestel J. & Uhl, J.R. 1977. Vector measures. Math Surveys, No. 15, Am. Math. Soc. Providence, R.I.

Khalil, R. 1980. Trace-class norm multipliers. Proc. Am. Math. Soc. **79:** 379–87.

Khalil, R. 1981. Pointwise multipliers. J. Univ. Kuwait (Sci.) **8:** 9–17.

Larsen, R. 1971. An introduction to the theory of multipliers. Springer-Verlag, Berlin.

Maddox, J. 1970. Elements of functional analysis. Cambridge University Press.

Schatten, R. 1960. Norm ideals of completely continuous operators. Springer-Verlag, Berlin.

Schur, J. 1911. Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. J. Reine Angew. Math. **140:** 1–28.

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ضوارب شور على دائرة الوحدة

رشدي خليل
قسم الرياضيات بجامعة الكويت

خلاصة

لنفرض أن T هي دائرة الوحدة وأن m هو قياس لبيك على T .
ضع $L^2(T)\hat{\otimes}L^2(T)$ ليكون هو حاصل الضرب التنسوري لـ $L^2(T,m)$ مع نفسها . ان التطبيق Φ المعرف على $T\times T$ يدعى ضاربا من نوع شور اذا كان $\Phi\cdot\psi$ ينتمي إلى $L^2(T)\hat{\otimes}L^2(T)$ لكل ψ في $L^2(T)\hat{\otimes}L^2(T)$ ، حيث $\Phi(x,y)\cdot\psi(x,y)=(\Phi\cdot\psi)\ (x,y)$
ان هدف هذا البحث هو دراسة ضوارب شور على $L^2(T)\hat{\otimes}L^2(T)$