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Prediction of cracking moment in reinforced concrete flexural members

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ABSTRACT

This paper presents an analytical method for determining the incipient-cracking resistance of reinforced concrete flexural members. The proposed method is based on the real stress–strain state of the constituent materials in the tension and compression zones at cracking. It is equally applicable to all common shapes of section and types of reinforcement. A graphical solution, intended to assist designers, has also been developed through a set of composite charts which serve all practical values of the parameters governing the cracking moment. A numerical example is given in Appendix B to illustrate the graphical procedure.

INTRODUCTION

Studies on the control of cracking in reinforced concrete flexural members usually recognize two stages: (1) incipient cracking, i.e. the first appearance of cracks, and (2) crack-opening, i.e. the formation and propagation of wide cracks.

Even well-designed reinforced concrete members are not free of cracks. However, only wide cracks, whose width exceeds specified permissible limits, are usually those considered detrimental to the integrity of the concrete and reinforcing steel (Gergely & Lutz 1968; Nawy 1968; CEB-FIP 1978; ACI Committee 224 1980).

Prediction of incipient cracking may be as important as control of crack width. There are instances where incipient cracks could damage or invalidate the service function of particular structures, such as fluid-retaining tanks, bins and silos. Accordingly, both the design and construction method of these structures should provide for an adequate control over such cracks. In design, this problem is circumvented by keeping the incipient-cracking resistance of a member not less than the external loading-induced effect (Hilal 1972; Yerlici 1975; ACI Committee 313 1977; Baikov & Sigalov 1981).

Moreover, with the recent widespread adoption of the ultimate strength method of design coupled with the use of high strength reinforcing steel and higher concrete strengths, the deflection of reinforced concrete members has become an increasingly important design concern. The deflection computations are based on the effective moment of inertia (for different sections along the span) in which the effects of the cracking moment and the applied moment are incorporated (Branson 1963; Burns & Siess 1966; Branson 1977; ACI Committee 318 1983).

In addition, determination of the crack spacing in a reinforced concrete member as well as computation of the member's stiffness warrant knowledge of the cracking moment, or strictly speaking, the elastic-plastic section modulus for the tension zone (Murashev *et al.* 1968).

As regards the cracking moment by which incipient-crack control for beams is effected, the current practice is to regard the beam as if it were entirely made of plain concrete and apply a nominal value for the concrete modulus of rupture, regardless of the size and relative proportions of the beam, as the ACI code provisions suggest (ACI Committee 435 1975; ACI Committee 318 1983). However, it is believed that it will be more rational to assess the cracking moment on the basis of the real stress–strain state (at the cracking stage) for both the concrete and reinforcing steel at a critical section.

In the analysis presented in this paper, due consideration is given to the experimentally established findings regarding the behaviour of reinforced concrete members, viz. (1) on approaching the cracking capacity of a reinforced concrete section, the concrete in the compression zone continues to behave elastically while in the tension zone inelastic deformations take place, and consequently the stress diagram therein becomes curved, and (2) the tension steel in particular and the compression steel as well contribute to the cracking resistance of the section (ACI Committee 318 1963; Branson 1963; Murashev *et al.* 1968).

The above discussion clearly indicates the need for proposing a reliable analysis procedure for predicting the early formation of the load-induced cracks taking into account the elastoplastic characteristics of the constituent materials, and in particular the actual extensibility of concrete. This constitutes the main objective of the present work. It should be mentioned that extensibility of a concrete is largely related to the concrete modulus of elasticity in tension, and, for this reason, the effective modulus of elasticity and not the conventional average secant modulus will be used in this approach.

COMPUTATION OF CRACKING MOMENT

To determine the cracking moment, M_{cr}, in the general case, let us examine an asymmetrical I-section doubly reinforced with tension steel, A_s, and compression steel, A_s', and subjected to an eccentric force, N, as shown in Fig. 1. This figure also

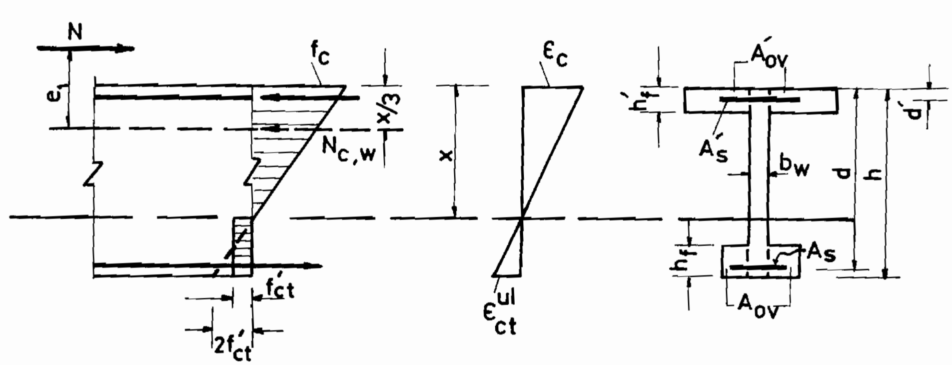

Fig. 1. Double-reinforced member at cracking. Typical forces, stress, strains, and dimensions.

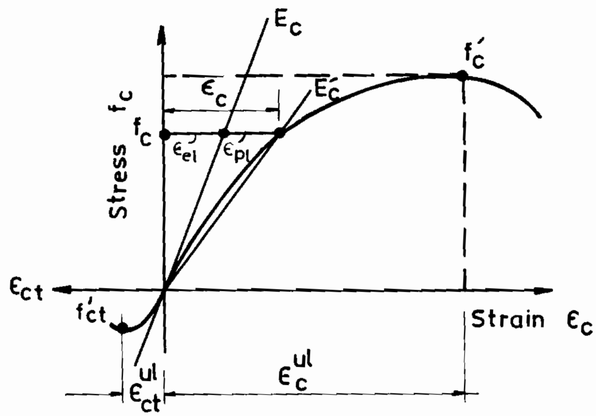

Fig. 2. Typical stress–strain diagram for concrete.

shows the stress- and strain-distribution diagrams at cracking. Fig. 2 represents a typical stress–strain diagram for concrete up to failure in both compression and tension. When a concrete specimen is subjected to a short-duration uniform axial loading, it undergoes a total strain composed of elastic and plastic parts. The initial tangent modulus of elasticity of concrete is denoted by E_c and refers only to the elastic strains produced by instantaneously applied loads.

The secant modulus of elasticity of concrete, E'_c, describes the total strain in compression, and may be related to E_c as follows (see Fig. 2):

$$f_c = \varepsilon_c E'_c = \varepsilon_{el} E_c \quad .$$

Thus

$$E'_c = (\varepsilon_{el}/\varepsilon_c) E_c = \nu E_c \quad . \tag{1}$$

The coefficient ν is equal to unity for elastic behaviour, and decreases with the increase of the stress–strength ratio and duration of loading.

Similarly, under tension

$$E'_{ct} = \nu_t E_c \quad . \tag{2}$$

When the tensile stress approaches the direct tensile strength, f'_{ct}, the average experimental value of ν_t, as obtained at NIIZhB, is 0·5 (Murashev *et al.* 1968).

Hence, ultimate tensile strain in concrete

$$\varepsilon_{ct}^{ul} = f'_{ct}/E'_{ct} = 2f'_{ct}/E_c \quad . \tag{3}$$

Denoting the depth of compression zone by x and the overall depth of the section by h, and referring to the strain diagram in Fig. 1 and to Equation (3) lead to

$$\varepsilon_c = \varepsilon_{ct}^{ul} \{x/(h - x)\} = (2f'_{ct}/E_c) \{x/(h - x)\} \tag{4}$$

where ε_c is the concrete strain at the extreme compression fibre. Recalling that the concrete strain in the compression zone at cracking is mostly elastic (hence $E'_c = E_c$) and $f_c = \varepsilon_c E'_c$, then Equation (4) can be rewritten as

$$f_c = 2f'_{ct}\{x/(h - x)\} \quad . \tag{5}$$

Equation (5) reveals that the line marking the stress distribution in the compression zone (Fig. 1), if produced downward, will meet the extreme fibre in tension at an imaginary stress of $2f'_{ct}$. This indicates that the concrete compressive stresses and steel stresses can easily be expressed in terms of twice the tensile strength, f'_{ct}.

Based on the above conclusion and taking moments of the internal forces (Fig. 1) about the position of the force resisted by the web in the compression zone, N_{cw}, yields

$$\begin{aligned} M_{cr} = f'_{ct}\{&b_w(h-x)(h/2+x/6) + A_{ov}(h-h_f/2-x/3) + 2(n-1)A_s(d-x/3) \\ &+ 2(n-1)A'_s(x-d')(x/3-d')/(h-x) \\ &+ 2A'_{ov}(x-h'_f/2)(x/3-h'_f/2)/(h-x)\} \quad . \end{aligned} \qquad (6)$$

Expressed in an abbreviated form, Equation (6) becomes

$$M_{cr} = f'_{ct}Z_{cr} \qquad (7)$$

where Z_{cr} is the elastic-plastic section modulus for the tension zone.

It is clear that Equation (6) involves one unknown, namely, the depth of the compression zone, x, which is determined from the condition that the external force N is equal to the sum of the internal forces perpendicular to the section through the member (Fig. 1). Thus

$$\begin{aligned} N = &f'_{ct}b_w x^2/(h-x) + 2f'_{ct}A'_{ov}(x-h'_f/2)/(h-x) + 2(n-1)f'_{ct}A'_s(x-d')/(h-x) \\ &- f'_{ct}b_w(h-x) - f'_{ct}A_{ov} - 2(n-1)f'_{ct}A_s \quad . \end{aligned} \qquad (8)$$

In the above equation, N is positive for eccentric compression, negative for eccentric tension, and zero for members in bending.

Multiplying by $(h-x)/f'_{ct}$ and transforming, then Equation (8) becomes

$$\begin{aligned} &(N/f'_{ct})h + b_w h^2 + A_{ov}h + 2(n-1)A_s h + 2(n-1)A'_s d' + A'_{ov}h'_f \\ &= (N/f'_{ct})x + 2b_w hx + A_{ov}x + 2(n-1)A_s x + 2(n-1)A'_s x + 2A'_{ov}x \quad . \end{aligned} \qquad (9)$$

Dividing Equation (9) by h and denoting x/h by k give

$$\begin{aligned} &N/f'_{ct} + b_w h + A_{ov} + 2(n-1)A_s + 2(n-1)A'_s d'/h + A'_{ov}h'_f/h \\ &= kN/f'_{ct} + 2kb_w h + kA_{ov} + 2(n-1)kA_s + 2(n-1)kA'_s + 2kA'_{ov} \quad . \end{aligned} \qquad (10)$$

Introducing $\delta' = d'/h$ and $\delta'_{ov} = 0{\cdot}5h'_f/h$, then Equation (10) may be written for k as

$$k = 1 - \frac{b_w h + 2(1-\delta'_{ov})A'_{ov} + 2(1-\delta')(n-1)A'_s}{2\{b_w h + A_{ov} + A'_{ov} + (n-1)A_s + (n-1)A'_s\} - A_{ov} + N/f'_{ct}} \quad . \qquad (11)$$

Examination of Equation (11) shows that, when $N = 0$ (case of simple bending), k may be assumed effectually equal to 0·5. It should be remembered, in particular, that A_s and A'_s are small quantities compared to the values of $b_w h$, and that the ratio $\delta' A'_s/b_w h$ is almost negligible.

For most practical cases of reinforced concrete beams, the following could be assumed at $k = 0{\cdot}5$ (that is at $x/h = 0{\cdot}5$):

(i) $(h - h_f/2 - x/3) = 0{\cdot}75h$
(ii) $2(d - x/3) = 1{\cdot}5h$
(iii) $d' = 0{\cdot}5h'_f \quad (=\delta' h)$.

Substituting the above values for the corresponding terms in Equation (6), the section modulus $Z_{cr} = M_{cr}/f'_{ct}$ may be expressed as

$$Z_{cr} = b_w h^2\{0{\cdot}292 + 0{\cdot}75\eta + 1{\cdot}5(n - 1)\varrho_1 + 4(0{\cdot}5 - \delta')(0{\cdot}167 - \delta')(\eta' + (n - 1)\varrho'_1)\} \quad (12)$$

where

$$\eta = A_{ov}/b_w h; \qquad \eta' = A'_{ov}/b_w h$$

$$\varrho_1 = A_s/b_w h; \qquad \varrho'_1 = A'_s/b_w h \; .$$

For a rectangular section, $\eta = \eta' = 0$; then replacing b_w by b gives

$$Z_{cr} = bh^2\{0{\cdot}292 + 1{\cdot}5(n - 1)\varrho_1 + 4(0{\cdot}5 - \delta')(0{\cdot}167 - \delta')(n - 1)\varrho'_1\} \; . \quad (13)$$

For a rectangular section with tension steel only

$$Z_{cr} = bh^2\{0{\cdot}292 + 1{\cdot}5(n - 1)\varrho_1\} \; . \quad (14)$$

For a T-section with tension steel only

$$Z_{cr} = b_w h^2\{0{\cdot}292 + 1{\cdot}5(n - 1)\varrho_1 + 4(0{\cdot}5 - \delta')(0{\cdot}167 - \delta')\eta'\} \; . \quad (15)$$

For an I-section with tension steel only

$$Z_{cr} = b_w h^2\{0{\cdot}292 + 0{\cdot}75\eta + 1{\cdot}5(n - 1)\varrho_1 + 4(0{\cdot}5 - \delta')(0{\cdot}167 - \delta')\eta'\} \; . \quad (16)$$

ANALYSIS USING CHARTS

Equation (12) represents the general expression for the elastic plastic section modulus pertaining to the tension zone of a beam at incipient cracking. It applies to all commonly shaped and reinforced beams. Equation (12) comprises numerous variables. Development of a graphical solution for Equation (12) would considerably facilitate its application by designers.

The ratio of $M_{cr}/(f'_{ct}b_w h^2)$ may be considered depending on five parameters: δ', η, η', $(n - 1)\varrho_1$ and $(n - 1)\varrho'_1$. The variable δ', as stated, is of the second degree. In practice, however, the terms containing the square of δ' can safely be neglected, and this is why the curves relating to these terms appear as straight lines.

The relationship between $M_{cr}/(f'_{ct}b_w h^2)$ on one side, and variables, η, η', $(n - 1)\varrho_1$ and $(n - 1)\varrho'_1$ on the other side is plotted in Fig. 3 and forms three sets of straight lines. As shown, four composite charts are constructed for four significant values of the coefficient δ'; interpolation can conveniently be made for other values of δ'. Every two composite charts are plotted overlapping each other, and solid and dashed lines are used to distinguish between them. For known values of δ', η, η', ϱ_1, ϱ'_1, as well as the concrete strength, the value of $M_{cr}/(f'_{ct}b_w h^2)$ can readily be determined from the appropriate charts of Fig. 3. The procedure for entering the successive sets of lines, in order to obtain finally the value of $M_{cr}/(f'_{ct}b_w h^2)$, is indicated on the charts by the broken arrow-headed line.

A numerical example, illustrating the procedure for determining the cracking moment, is given in Appendix B.

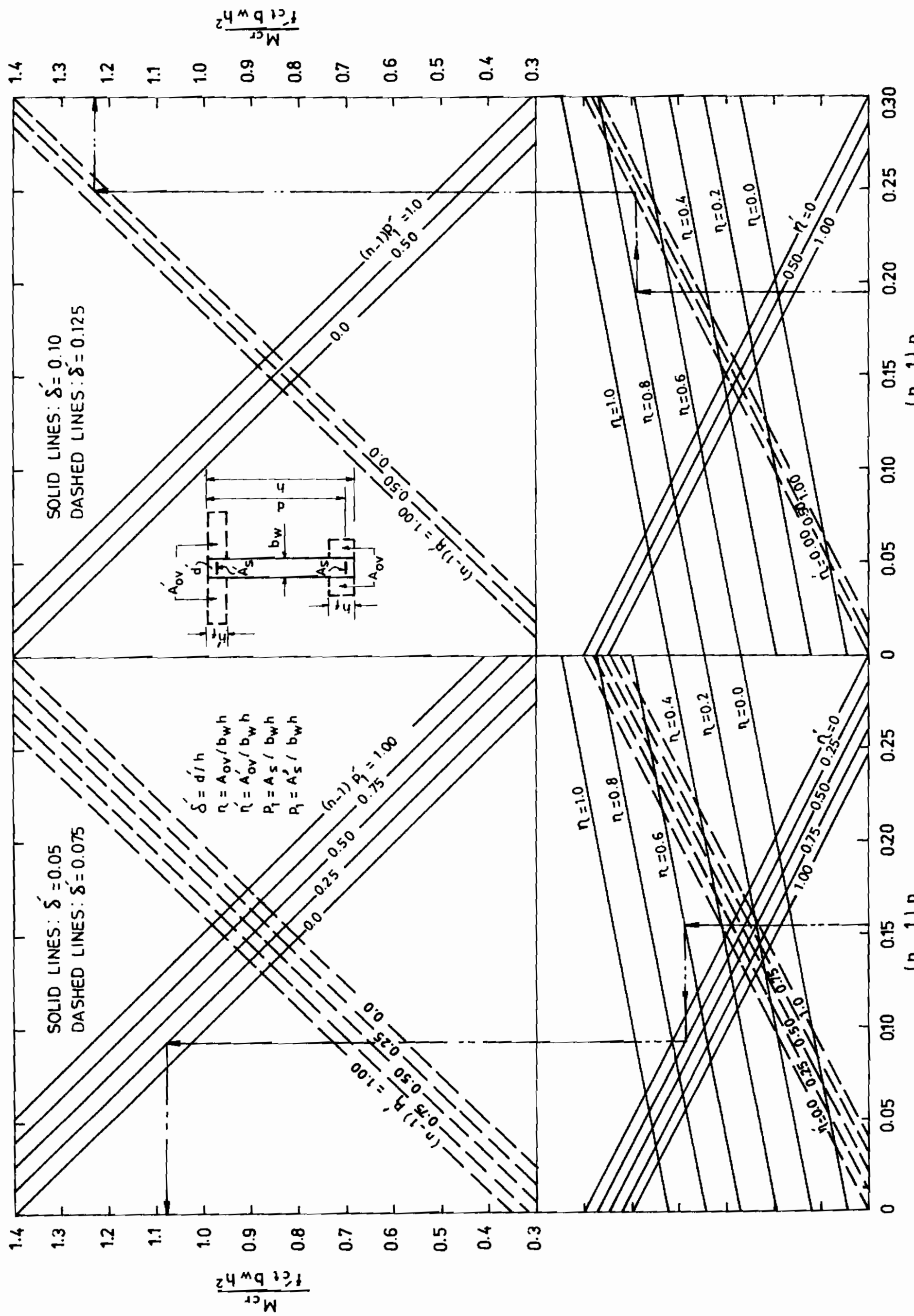

Fig. 3. $M_{cr}/f'_{ct}b_wh^2$ versus $(n-1)\varrho_1$ curves for different values of coefficients δ', η, η' and ϱ'_1.

SUMMARY AND CONCLUSIONS

A method for predicting the development of incipient cracks in flexural members, with a wide variety of shapes and reinforcements, has been proposed. The method is portrayed by a single general formula [Equation (12)] which permits a consistent assessment of the cracking moment. The latter is mainly dependant on the concrete area, amount of tension reinforcement, concrete strength and its initial modulus of elasticity, total depth of beam, and to a lesser degree, on the shape of the cross-section, and amount and position of compression reinforcement.

A graphical representation of Equation (12) has also been presented. The charts developed provide designers with a faster and almost equally accurate solution. They allow interpolation for all practical values of the coefficient $\delta' = d'/h$. Being dimensionless, the charts can be used with any system of units.

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APPENDIX A. RECOMMENDED PROVISIONS FOR E_c, E_s, f_{ct}' AND n

According to experimental data available from different sources, the initial tangent modulus of elasticity for moist-cured concrete may be given by the emperical formula

$$E_c = w_c^{1\cdot5} 0{\cdot}86 f_c'/(27 + 1{\cdot}75 f_c'), \quad \text{N/mm}^2$$

where w_c is the density of concrete, kg/m^3, and f_c' is the specified cylinder compressive strength in N/mm^2.

For normal-density concrete,

$$E_c = 96{,}000 f_c'/(27 + 1{\cdot}75 f_c') \quad .$$

The modulus of elasticity for all nonprestressed steel may be taken as

$$E_s = 200{,}000, \quad \text{N/mm}^2 \quad .$$

The direct tensile strength for normal-weight concrete is about 88% to 90% of the split-cylinder tensile strength and, consequently, may be taken as

$$f_{ct}' = 0{\cdot}48\sqrt{f_c'}, \quad \text{N/mm}^2 \quad \text{(Wang \& Salmon (1985).}$$

The values of these parameters in the usual strength range of $f_c' = 15$ to $40\,\text{N/mm}^2$ are presented (slightly modified) in Table 1.

Table 1

f_c', N/mm^2	E_c, N/mm^2	f_{ct}', N/mm^2	$n = E_s/E_c$
15	27,000	1·86	7·4
20	30,900	2·15	6·5
25	33,900	2·40	5·9
30	36,300	2·63	5·5
35	38,100	2·84	5·25
40	39,600	3·04	5·0

APPENDIX B. EXAMPLE OF CRACK-CONTROL PROCEDURE

Determine the cracking moment and examine the incipient crack control situation for a ribbed floor panel (T-section) with the dimensions shown in Fig. 4 and having the following data: design span = 6 m, service loads (including own weight of slab and joist) = 600 kg/m, and $f_c' = 20\,\text{N/mm}^2$.

Solution. Read $n = 6{\cdot}5$, $f_{ct}' = 2{\cdot}15\,\text{N/mm}^2$ and $A_s = 508\,\text{mm}^2$.

$$\delta' = 0{\cdot}5 \times 60/350 = 0{\cdot}086 > 0{\cdot}075$$
$$< 0{\cdot}10$$

$$(n - 1)\varrho_1 = (6{\cdot}5 - 1) \times \frac{508}{150 \times 350} = 0{\cdot}053$$

$$\eta' = \frac{60(750 - 150)}{150 \times 350} = 0{\cdot}686 \quad .$$

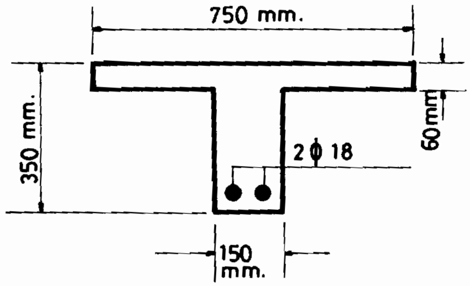

Fig. 4. Dimensions of the T-section.

Enter, in Fig. 3, the chart designated by $\delta' = 0{\cdot}075$ with the calculated value of $(n - 1)\varrho_1$, proceed vertically to the line for $\eta = 0$, and then horizontally to the line for $\eta' = 0{\cdot}686$ (by interpolation), then move vertically to the line of $(n - 1)\varrho_1' = 0$, and finally proceed horizontally to read $M_{cr}/(f_{ct}'b_w h^2) = 0{\cdot}476$. Enter the chart corresponding to $\delta' = 0{\cdot}10$ and proceed in the same manner as above to read finally $M_{cr}/(f_{ct}'b_w h^2) = 0{\cdot}445$.

The value of $M_{cr}/f_{ct}'b_w h^2$ corresponding to $\delta' = 0{\cdot}086$ may be determined by interpolating between the above two values. Thus

$$M_{cr}/(f_{ct}'b_w h^2) = 0{\cdot}445 + (0{\cdot}476 - 0{\cdot}445) \times \frac{0{\cdot}014}{0{\cdot}025} = 0{\cdot}462 \quad .$$

Then

$$M_{cr} = 0{\cdot}462 \times 2{\cdot}15 \times 150 \times 350^2/10^6 = 18{\cdot}25\,\text{kN m}$$

$$M = \left(\frac{600 \times 6^2}{8} \times 9{\cdot}81\right)\Big/10^3 = 26{\cdot}49\,\text{kN m} > M_{cr} \quad .$$

Therefore, incipient cracks will develop due to service loads.

APPENDIX C. NOTATION

A_{ov} = cross-sectional area of flange overhangs in the tension zone
A_{ov}' = cross-sectional area of flange overhangs in the compression zone
A_s = area of nonprestressed tension reinforcement
A_s' = area of compression reinforcement
b = width of a rectangular section
b_w = width of web
d = distance from extreme compression fibre to centroid of tension reinforcement
d' = distance from extreme compression fibre to centroid of compression reinforcement
e_1 = eccentricity of the longitudinal load N measured from the internal compressive force resisted by the web
E_c = initial tangent modulus of elastiticy of concrete in compression and tension
E_c' = secant modulus of elasticity of concrete in compression
E_{ct}' = secant modulus of elasticity of concrete in tension
E_s = modulus of elasticity of steel

f_c = stress in concrete at extreme compression fibre
f_c' = specified cylinder compressive strength of concrete
f_{ct}' = direct tensile strength of concrete
h = overall depth of member
h_f = thickness of tension flange
h_f' = thickness of compression flange
k = ratio of depth of compression zone to overall depth of member = x/h
M = maximum moment in member due to externally applied service loads
M_{cr} = cracking moment
n = modular ratio of elasticity = E_s/E_c
N = service eccentric load normal to the cross section
N_{cw} = internal compressive force resisted by the web
w_c = unit mass of concrete = 2300 kg/m^3 for normal-weight concrete
x = depth of compression zone before the advent of cracking
Z_{cr} = elastic-plastic section modulus for tension zone = M_{cr}/f_{ct}'
δ' = d'/h
δ_{ov}' = $0{\cdot}5h_f'/h$
ε_c = strain in concrete at extreme compression fibre
ε_{ct}^{ul} = ultimate tensile strain in concrete
η = $A_{ov}/b_w h$
η' = $A_{ov}'/b_w h$
ν_c = modular ratio of elasticity of concrete in compression = E_c'/E_c
ν_t = modular ratio of elasticity of concrete in tension = E_{ct}'/E_c
ϱ_1 = ratio of tension reinforcement = $A_s/b_w h$
ϱ_1' = ratio of compression reinforcement = $A_s'/b_w h$

التنبؤ بعزم التشقق
في القطع الخرسانية المسلحة

محفوظ سعيد الريس
قسم الهندسة المدنية بجامعة الكويت

خلاصة

يقدم هذا البحث طريقة تحليلية لتعيين مدى مقاومة قطع الانحناء الخرسانية المسلحة للشقوق الأولية . وهذه الطريقة مبنية على حالة الإجهاد والإنفعال الفعلية التي تكون عليها المواد المكونة للقطعة في منطقتي الشد والضغط عند حدوث التشقق . هذا وتصلح الطريقة لجميع الأشكال المألوفة للمقاطع ولجميع أنواع التسليح .

كما يشمل البحث طريقة بيانية استنبطت عبر مجموعة رسوم مركبة تخدم كل القيم العملية للعوامل التي تحكم عزم التشقق وذلك بغية مساعدة مهندسي التصميم . وفي الملحق ب أعطي مثال عددي لتوضيح استخدام الطريقة البيانية .