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Some theorems on the structure of Liapunov stable motions

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ABSTRACT

Since Liapunov first introduced his famous definition of the concept of stability of motion, many definitions of stability have been formulated which are related to that of Liapunov. This paper is devoted to establishing new relationships between these definitions.

INTRODUCTION

We first give the basic definitions and notations used in this paper. We shall denote the real numbers and nonnegative real numbers by $\mathbb{R}$ and $\mathbb{R}^+$, respectively.

A triple $(X, \mathbb{R}, \pi)$ consisting of a metric space X with metric ρ, the set $\mathbb{R}$ and a continuous mapping from the product space $X \times \mathbb{R}$ into X is called a dynamical system or a (continuous) flow whenever π satisfies the following axioms:

1. $\pi(x, 0) = x$ for every $x \in X$ (identity axiom),
2. $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$ for every $x \in X$ and $t_1, t_2 \in \mathbb{R}$ (group axiom).

The space X is called the phase space. In the sequel we shall denote $\pi(x, t)$ by xt for brevity. For each $x \in X$, $x\mathbb{R}^+$ is called the positive semi-trajectory of x. The point x is said to be positively Lagrange stable if the closure $\overline{x\mathbb{R}^+}$ is a compact set. The set $D^+(x) = \{y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \to x$ and $x_n t_n \to y\}$ are called the first positive prolongations of x. When X is locally compact and $X^* = X \cup \{\infty\}$ is its one-point compactification, for each $x \in X$ we define the set $D_*^+(x)$ by $D_*^+(x) = \{y \in X^*$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \to x$ and $x_n t_n \to y\}$ (see Bhatia & Szegö (1970)).

A point $x \in X$ is called positively Liapunov stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\rho(xt, yt) < \varepsilon$ provided $\rho(x, y) < \delta$ and $t \in R^+$. A point x in a locally compact metric space X is called positively weakly Liapunov stable whenever $D_*^+(x, x) \subseteq \Delta_*$, where Δ_* denotes the diagonal in $X^* \times X^*$ (Elaydi 1982). The following is a new definition: a point $x \in X$ is said to be positively semi-Liapunov stable provided $D^+(x, x) \subseteq \Delta$, where Δ is the diagonal of $X \times X$ (Ahmad 1970).

A point $x \in X$ is said to be of characteristic 0^+ if $D^+(x) = \overline{x\mathbb{R}^+}$ (Ahmad 1970). It is called a point of strong characteristic 0^+ if given sequences $\{x_n\}$ in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \to x$ and $x_n t_n \to y \in X$, then $xt_n \to y$ (Elaydi & Kaul 1982).

Finally, the point $x \in X$ is said to be positively even continuous if for each $y \in X$ and each neighborhood U of y, there is a neighborhood V of x and a neighborhood W of y such that $Vt \subseteq U$ whenever $xt \in W$ for some $t \in \mathbb{R}^+$.

MAIN THEOREMS

To begin with, we introduce the following lemma which characterizes the positively even continuous points and which we shall use several times in this paper. This lemma was given as an exercise in Kelley (1955).

Lemma 1. A necessary and sufficient condition for a point $x \in X$ to be positively even continuous is that whenever there are sequences $\{x_n\}$ in X, $\{t_n\}$ in $\mathbb{R}^+$ such that $x_n \rightarrow x$ and $xt_n \rightarrow y$, then $x_n t_n \rightarrow y$.

Theorem 1. If a point $x \in X$ is positively Liapunov stable, then it is positively semi-Liapunov stable.

Proof. Let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $\rho(xt, \xi t) < \varepsilon/2$ provided $\rho(x, \xi) < \delta$ and $t \in \mathbb{R}^+$. Let $(y, z) \in D^+(x, x)$. Hence, there is a sequence $\{(y_n, z_n)\}$ in $X \times X$ and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $(y_n, z_n) \rightarrow (x, x)$ and $(y_n, z_n)t_n \rightarrow (y, z)$. In other words, there are sequences $\{y_n\}$, $\{z_n\}$ in X and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $y_n \rightarrow x$, $z_n \rightarrow x$, $y_n t_n \rightarrow y$ and $z_n t_n \rightarrow z$. Therefore, there is a positive integer N such that $\rho(y_n, x) < \delta$ and $\rho(z_n, x) < \delta$ whenever $n \geqslant N$. Hence $\rho(y_n t_n, xt_n) < \varepsilon/2$ and $\rho(z_n t_n, xt_n) < \varepsilon/2$ for $n \geqslant N$, and these inequalities imply that whenever $n \geqslant N$, then $\rho(y_n t_n, z_n t_n) < \varepsilon$. Passing in this inequality to the limit as $n \rightarrow \infty$, we find $\rho(y, z) \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $y = z$, i.e. $(y, z) \in \Delta$. As (y, z) is an arbitrary element of $\rho^+(x, x)$, $D^+(x, x) \subseteq \Delta$. The theorem is proved.

We next indicate the relationship between positively Liapunov stable points and positively weakly Liapunov stable points.

Theorem 2. If x is a positively Liapunov stable point in a locally compact metric space X, then it is positively weakly Liapunov stable (Elaydi 1982).

The following two theorems show how the concept of being positively Liapunov stable is gradually generalized to the concepts of having strong characteristic 0^+ and characteristic 0^+, respectively.

Theorem 3. If a point $x \in X$ is positively Liapunov stable, then it is of strong characteristic 0^+ (Elaydi & Kaul 1982).

Theorem 4. If a point $x \in X$ is of strong characteristic 0^+, then it is of characteristic 0^+ (Elaydi & Kaul 1982).

Theorem 5. If $x \in X$ is a positively Liapunov stable point, then it is positively even continuous.

Proof. By virtue of the positive Liapunov stability of x, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\rho(xt, \xi t) < \varepsilon/2$ whenever $\rho(x, \xi) < \delta$ and $t \in \mathbb{R}^+$. Let $\{x_n\}$ be a sequence in X and $\{t_n\}$ in $\mathbb{R}^+$ such that $xt_n \rightarrow y \in X$. Then there is a positive integer N_1 such that

$\rho(x_n, x) < \delta$ whenever $n \geqslant N_1$. Therefore $\rho(x_n t_n, xt_n) < \varepsilon/2$ for $n \geqslant N_1$. Also, it follows from $xt_n \to y$ that there is a positive integer N_2 such that $\rho(xt_n, y) < \varepsilon/2$ provided $n \geqslant N_2$. Let $N = \max\{N_1, N_2\}$, then if $n \geqslant N$,

$$\rho(x_n t_n, y) \leqslant \rho(x_n t_n, xt_n) + \rho(xt_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, $x_n t_n \to y$. This shows that by using Lemma 1, x is positively even continuous.

The following three theorems are partial converses of Theorems 3, 4 and 6.

Theorem 6. If a positively weakly Liapunov stable point of a locally compact metric space X is positively Lagrange stable, then it is positively Liapunov stable.

Proof. Suppose that x is not positively Liapunov stable. Then there is an $\varepsilon_0 > 0$ such that no matter what $\delta > 0$ is, there is a $t \in \mathbb{R}^+$ and a point $\xi \in X$ such that $\rho(\xi, x) < \delta$, whereas $\rho(\xi t, xt) \geqslant \varepsilon_0$. Take an arbitrary sequence of positive numbers $\{\delta_n\}$ such that $\delta_n \to 0$. Then for any natural number n, there exist $t_n \in \mathbb{R}^+$ and $x_n \in X$ such that $\rho(x_n, x) < \delta_n$ and $\rho(x_n t_n, xt_n) \geqslant \varepsilon_0$. It is clear that $x_n \to x$. Since the sequence $\{xt_n\}$ is contained in the compact set $\overline{x\mathbb{R}^+}$, it contains a convergent subsequence. In order not to complicate the notation, assume that the sequence $\{xt_n\}$ itself converges. Let $xt_n \to y \in X$. Similarly, the sequence $\{x_n t_n\}$ can be assumed to be convergent in the compact space X^*. Let $x_n t_n \to z \in X^*$. Now, $\{(x_n, z)\}$ is a sequence in $X \times X$ and $\{t_n\}$ in $\mathbb{R}^+$ such that $(x_n, x) \to (x, x) \in X \times X$ and $(x_n, x)t_n = (x_n t_n, xt_n) \to (z, y) \in X^* \times X^*$. Hence, $(z, y) \in D_*^+(x, x)$. As $D_*^+(x, x) \subset \Delta_*$ due to the positive weak Liapunov stability of x, $(z, y) \in \Delta_*$, i.e. $z = y$. Since $y \in X$, $z \in X$. Passing in $\rho(x_n t_n, xt_n) \geqslant \varepsilon_0$ to the limit as $n \to \infty$, we find $\rho(y, y) \geqslant \varepsilon_0$, i.e. $0 \geqslant \varepsilon_0$. The contradiction obtained proves the theorem.

Theorem 7. If a point is of strong characteristic 0^+ in a compact metric space X, then it is positively Liapunov stable.

Proof. We shall prove the theorem by assuming the contrary and arriving at a contradiction. So let x not be positively Liapunov stable, then there is an $\varepsilon_0 > 0$ such that for each natural number n, there exist $t_n \in \mathbb{R}^+$ and an $x_n \in X$ such that $\rho(x_n, x) < 1/n$ and $\rho(x_n t_n, xt_n) \geqslant \varepsilon_0$. Since X is compact, the sequence $\{x_n t_n\}$ may be assumed to be convergent to a point $y \in X$. But then $xt_n \to y$ by virtue of the fact that x is of strong characteristic 0^+. Passing in $\rho(x_n t_n, xt_n) \geqslant \varepsilon_0$ to the limit as $n \to \infty$, we arrive at the inequality $\rho(y, y) \geqslant \varepsilon_0$, i.e. $0 \geqslant \varepsilon_0$. This contradiction proves the theorem.

Theorem 8. If x is positively even continuous and positively Lagrange stable, then it is positively Liapunov stable.

Proof. Using Lemma 1, the proof is quite similar to that of Theorem 7. Here the sequence which is assumed to be convergent is $\{xt_n\}$ due to the Lagrange stability of x. The convergence $xt_n \to y$ implies that $x_n t_n \to y$ by virtue of the fact that x is positively even continuous.

Thus far, we have been studying connections between the concept of positive Liapunov stability and other notions related to it. We now turn to investigating interrelationships between these notions.

Theorem 9. If x is a positively weakly Liapunov stable point in a locally metric space X, then it is positively semi-Liapunov stable.

Proof. From the definitions of D^+ and D_*^+, it is clear that $D^+(x, x) \subseteq D_*^+(x, x)$. Since $D_*^+(x, x) \subseteq \Delta_* = \Delta \cup \{(\infty, \infty)\}$ due to the fact that x is positively weakly Liapunov stable, it follows that $D^+(x, x) \subseteq \Delta_*$. Taking into consideration that $D^+(x, x)$ is a subset of $X \times X$, we conclude that $D^+(x, x) \subseteq \Delta$. Hence, the point x is positively semi-Liapunov stable.

Theorem 10. If $x \in X$ is of strong characteristic 0^+, then it is positively semi-Liapunov stable.

Proof. Let $(y, z) \in D^+(x, x)$ for some $y, z \in X$. Then there is a sequence $\{(y_n, z_n)\}$ in $X \times X$ and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $(y_n, z_n) \to (x, x)$ and $(y_n, z_n)t_n \to (y, z)$. Since x is of strong characteristic 0^+, $xt_n \to y$ and $xt_n \to z$. Hence, $y = z$, i.e. $(y, z) \in \Delta$. As (y, z) is an arbitrary element of $D^+(x, x)$, we conclude that $D^+(x, x) \subseteq \Delta$, and the theorem is proved.

Theorem 11. If x is a point of strong characteristic 0^+ and positively even continuous in a locally compact metric space X, then x is positively weakly Liapunov stable.

Proof. Let $(y, z) \in D_*^+(x, x)$ for some $y, z \in X^*$. If $(y, z) = (\infty, \infty)$, then $(y, z) \in \Delta_* = \Delta \cup \{(\infty, \infty)\}$. If $(y, z) \in X \times X$, then, clearly, $(y, z) \in D^+(x, x)$. Taking into consideration that x, by virtue of Theorem 10, is positively semi-Liapunov stable, we conclude that $(y, z) \in \Delta \subseteq \Delta^*$. To complete the proof of the theorem we need show that it is impossible to have one of the points y, z belong to X whereas the other does not. Assume without loss of generality that $y \in X$ and $z = \infty$. Then $(y, \infty) \in D_*^+(x, x)$ implies that there is a sequence $\{(y_n, z_n)\}$ in $X \times X$ and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $(y_n, z_n) \to (x, x)$ and $(y_n, z_n)t_n \to (y, \infty)$. Since x is of strong characteristic 0^+, it follows that $xt_n \to y$. Now, since $z_n \to x$ and $xt_n \to y$, it follows from the positive even continuity of x that $z_n t_n \to y$ according to Lemma 1, which is a contradiction. We conclude from the above argument that if $(y, z) \in D_*^+(x, x)$ for some $y, z \in X^*$, then either $y = z = \infty$ or $y = z \in X$. Hence, $D_*^+(x, x) \subseteq \Delta_*$ and the theorem is completely proved.

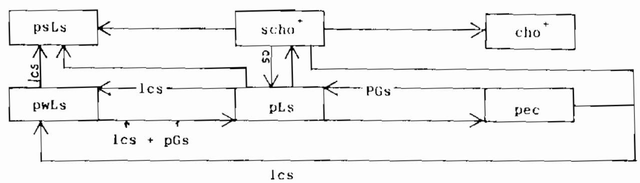
Finally, we have the following theorem.

Theorem 12. If $x \in X$ is of strong characteristic 0^+ in the one point compactification space X^* of the locally compact space X, then x is positively weakly Liapunov stable.

Proof. Let $(y, z) \in D_*^+(x, x)$ for some $y, z \in X^*$. Then there is a sequence $\{(y_n, z_n)\}$ in $X \times X$ and a sequence $\{t_n\}$ in $\mathbb{R}^+$ such that $(y_n, z_n) \to (x, x)$ and $(y_n, z_n)t_n \to (y, z)$. Since x is of strong characteristic 0^+ in X^*, $xt_n \to y$ and $xt_n \to z$. Hence, $y = z$, i.e. $(y, z) \in \Delta^*$. Since (y, z) is an arbitrary element of $D_*^+(x, x)$, it follows that $D_*^+(x, x) \subseteq \Delta^*$, i.e. x is positively weakly Liapunov stable.

Remark. Negative versions of the above theorems may be given in an obvious way.

The following diagram depicts schematically the above implications.



NOTATION

cho^+	x is of characteristic 0^+
$scho^+$	x is of strong characteristic 0^+
pec	x is positively even continuous
pLs	x is positively Liapunov stable
psLs	x is positively semi-Liapunov stable
pwLs	x is positively weakly Liapunov stable
cs	in a compact metric space
lcs	in a locally compact metric space
pGs	if x is positively Lagrange stable

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بعض المبرهنات حول بنية الحركات المستقرة بمفهوم ليبونوف

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خلاصة

منذ أن قدم ليبونوف لأول مرة تعريفه الشهير لمفهوم استقرار الحركة ، تمت صياغة عدة تعاريف للاستقرار مرتبطة بتعريف ليبونوف . وهذا البحث مكرس لإيجاد علاقات جديدة بين هذه التعاريف .