*J. Univ. Kuwait (Sci.)* **14**, 1987

# Almost contact λ-manifolds

ADNAN AL-AQEEL

*Department of Mathematics, University of Kuwait, P.O. Box 5969, Safat 13060, Kuwait*

## ABSTRACT

In this paper, a generalization of almost contact manifolds is introduced. A tensor $S$ of type (1, 1) which corresponds to almost contact manifold structure tensor $F$, is defined. Finally the construction of an almost contact $\lambda$-manifold is found.

## INTRODUCTION

Let $M$ be a $c^\infty$, $n$-dimensional manifold. Let $S$ be a $c^\infty$ tensor field of type (1, 1) on $M$. Suppose that

$$f(s) = \prod_{k=1}^{r} g_k(s)$$

where $g_k(s)$, $k = 1, 2, \ldots, r$ are relatively prime polynomials. If $f(S) = 0$ then at each point $P \in M$, we have

$$M_p = \bigoplus M_{pi}, \quad i = 1, \ldots, r \quad \text{(direct sum)}$$

and

$$M_{pi} = \text{kernel } g_i(S) \quad \text{(Hoffman \& Kunze 1963, p. 179).}$$

This means that we are going to have $r$-differentiable distributions $M_1, \ldots, M_r$ on $M$ such that the distribution $M_i$ assigns $M_{pi}$ at $P \in M$.

Suppose that on each distribution $M_k, k = 1, \ldots, r$ we have a $c^\infty$ vector field $U_k$ and a 1-form $u^k$ such that

$$SU_k = \lambda_k U_k, \quad \lambda_k \neq \pm 1 \text{ is a scalar} \tag{1}$$

$$H^2 + KI = Ku^k \otimes U_k \quad \text{where } H = (S - \lambda_k I) \text{ and } K = 1 - \lambda_k^2. \tag{2}$$

$M$ will be called almost contact $\lambda$-manifold, and $\{S, u^k, U_k, k = 1, \ldots, r\}$ will be called almost contact $\lambda$-structure.

---

This research was done while the author was spending his sabbatical leave in Claremont Graduate School, Claremont Colleges, California, U.S.A.

## SOME THEOREMS ON ALMOST CONTACT λ-MANIFOLDS

*Theorem 1.* On an almost contact $\lambda$-manifold we have

$$\text{(i)}\ u^k(U_k) = 1,$$
$$\text{(ii)}\ u^k \circ S = \lambda_k u^k.$$

*Proof.* (i) From Equation (2) we have

$$H^2 U_k + kU_k = ku^k(U_k)U_k.$$

But $HU_k = 0$, from (1)

$$\therefore \quad 0 + KU_k = Ku^k(U_k)U_k \Rightarrow u^k(U_k) = 1.$$

(ii) Pre- and post-multiply (1) by $H$; we get

$$H^3 + KH = Ku^k \circ HU_k = 0 = K(u^k \circ H) \otimes U_k$$

since $HU_k = 0$

$$\therefore \quad u^k \circ H = 0 \Rightarrow u^k \circ (S - \lambda_k I) = 0 \Rightarrow u^k \circ S = \lambda_k u^k. \quad //$$

*Theorem 2.* On an almost contact $\lambda$-manifold $\lambda_k$ and $\lambda_k \pm i\sqrt{K}$ are the eigenvalues of $S$, and $\lambda_k$ is of multiplicity 1 $(i = \sqrt{-1})$.

*Proof.* Suppose that $a$ is an eigenvalue of $S$ corresponding to the eigenvector $X \in M_k$. Then we have

$$(S - \lambda_k I)^2 X + KX = Ku^k(X)U_k \Rightarrow [(a - \lambda_k)^2 + 1 - \lambda_k^2]X = Ku^k(X)U_k$$
$$\Rightarrow (a^2 - 2a\lambda_k + 1)X = Ku^k(X)U_k. \tag{3}$$

(i) $X = bU_k$, $b$ is a scalar

$$\therefore \quad b(a^2 - 2a\lambda_k + 1)U_k = Ku^k(bU_k)U_k = bKU_k$$
$$\therefore \quad a^2 - 2a\lambda_k + 1 = 1 - \lambda_k^2 \Rightarrow (a - \lambda_k)^2 = 0 \Rightarrow a = \lambda_k$$

and $\lambda_k$ is of multiplicity 1, since $SU_k = \lambda_k U_k$ is the only vector.

(ii) $X$ and $U_k$ are linearly independent

$$\therefore \quad a^2 - 2a\lambda_k + 1 = 0 \Rightarrow a = \lambda_k \pm iK. \quad //$$

*Theorem 3.* Let $M^n$ be an almost contact $\lambda$-manifold, then it contains a tangent bundle $\Gamma$ of dimension $r_1$, a tangent bundle $\pi$ of dimension $r_2$, and a tangent bundle $\bar{\pi}$ of dimension $r$, such that

$$n = r_1 + r_2 + r, \quad \pi \cap \Gamma = \phi, \quad \pi \cap \bar{\pi} = \phi, \quad \Gamma \cap \bar{\pi} = \phi$$

and

$$\pi \cup \Gamma \cup \bar{\pi} = \text{tangent bundle of } M.$$

*Proof.* On $M_k$, let $P_x, x = 1, \ldots, d$ and $Q_y, y = 1, \ldots, \bar{d}$ be linearly independent vectors corresponding to the eigenvalues $\lambda_k \pm iK$. Let

$$a^x P_x + b^y Q_y + cU_k = 0 \tag{4}$$
$$\Rightarrow a^x SP_x + b^y SQ_y + cSU_k = 0. \tag{5}$$

Multiply (4) by $-\lambda_k$ and add to (5), we get

$$(a^x iK)P_x - (b^y iK)Q_y = 0 \Rightarrow a^x P_x - b^y Q_y = 0. \tag{6}$$

Effect (6) by $S$, we get

$$a^x(\lambda_k + iK)P_x - b^y(\lambda_k - iK)Q_y = 0 \Rightarrow a^x P_x + b^y Q_y = 0. \tag{7}$$

From (4), (6) and (7) we have $P_x$, $Q_y$ and $U_k$ are linearly independent. Let

$$-2Kl_k = H^2 + i\sqrt{K}H, \quad -2Km_k = H^2 - i\sqrt{K}H,$$

$$Kn_k = H^2 + KI = Ku^k \otimes U_k, \quad i = \sqrt{-1}.$$

It is easy to check that

$$l_k P_x = P_x, \quad l_k Q_y = 0 \quad \text{and} \quad l_k U_k = 0$$

$$m_k P_x = 0, \quad m_k Q_y = Q_y \quad \text{and} \quad m_k U_k = 0$$

$$n_k P_x = 0, \quad n_k Q_y = 0 \quad \text{and} \quad n_k U_k = U_k.$$

This means that on $M_k$ we have tangent sub-bundles $\pi_k$, $\Gamma_k$ and a line $\bar{\pi}_k$ corresponding to the projections $l_k$, $m_k$ and $n_k$ respectively. Also

$$\pi_k \cap \Gamma_k = \phi, \quad \pi_k \cap \bar{\pi}_k = \phi, \quad \Gamma_k \cap \bar{\pi}_k = \phi,$$

and

$$M_k = \pi_k \cup \Gamma_k \cup \bar{\pi}_k.$$

Let

$$\pi = \bigoplus \pi_k, \quad \Gamma = \bigoplus \Gamma_k \quad \text{and} \quad \bar{\pi} = \bigoplus \bar{\pi}_k \quad \text{(direct sum)}.$$

Then $\pi$, $\Gamma$ and $\bar{\pi}$ are tangent bundles on $M$ satisfying

$$\pi \cap \Gamma = \phi, \quad \Gamma \cap \bar{\pi} = \phi, \quad \pi \cap \bar{\pi} = \phi,$$

and

$$\pi \cup \Gamma \cup \bar{\pi} = \text{tangent bundle of } M.$$

If $r_1$ = dimension of $\pi$, $r_2$ = dimension of $\Gamma$, then

$$n = r_1 + r_2 + r. \quad //$$

## CONSTRUCTION OF AN ALMOST CONTACT λ-MANIFOLD

We shall now give the converse of Theorem 3. Let $M$ be an $n$-dimensional manifold. Let $S$ be a $c^\infty$ tensor field of type (1, 1) on $M$. Suppose that

$$f(s) = \prod_{k=1}^{r} g_k(s)$$

where $g_k(s)$ are relatively prime polynomials such that $f(S) = 0$. Then we are going to have $r$-differentiable distributions $M_1, \ldots, M_r$ on $M$, where

$$M_k = \text{kernel } g_k(S) \quad \text{and} \quad SM_k \subset M_k.$$

*Theorem 4.* Let $M$ be as above, suppose that the eigenvalues of $S$ are $\lambda_k$, $\lambda_k \pm iK$, and that on the distribution $M_k$ we have tangent sub-bundles $\pi_k$, $\Gamma_k$ of dimensions $t_1$ and $t_2$ respectively, corresponding to the eigenvalues $\lambda_k \pm iK$ and a line $\bar{\pi}$ corresponding to the eigenvalue $\lambda_k$ such that

$$\pi_k \cap \Gamma_k = \varnothing, \quad l_k \cap \bar{\pi}_k = \varnothing, \quad \pi_k \cap \bar{\pi}_k = \varnothing$$

$$\pi_k \cup \Gamma_k \cup \bar{\pi}_k = M_k.$$

Then $M$ is an almost contact $\lambda$-manifold.

*Proof.* Let $P_x$ be $t_1$ linearly independent eigenvectors in $\pi_k$, $Q_y$ be $t_2$ linearly independent eigenvectors in $\Gamma_k$, and $U_k$ be a vector in $\bar{\pi}_k$. $\{P_x, Q_y, U_k\}$ span $M_k$ and they are linearly independent. Let $\{P^x, Q^y\}$ be the dual basis of $\{P_x, Q_y\}$, and let $u^k$ be a 1-form such that $u^k(U_k) = 1$

$$I = p^x \otimes P_x + Q^y \otimes Q_y + u^k \otimes U_k. \tag{8}$$

Define

$$(S - \lambda_k I) = H = iK[P^x \otimes P_x - Q^y \otimes Q_y] \tag{9}$$

$$(S - \lambda_k I)^2 = H^2 = iK[P^x \otimes HP_x - Q^y \otimes HQ_y] = iK[P^x \otimes P_x + Q^y \otimes Q_y]$$

$$= -K[I - u^k \otimes U_k]$$

$$\therefore \quad H^2 + KI = Ku^k \otimes U_k.$$

From (9) we also have

$$SU_k = \lambda_k U_k.$$

Hence $M$ is an almost contact $\lambda$-manifold. //

*Theorem 5.* If $\lambda_k^2 - 1 < 0$, in an almost contact $\lambda$-manifold $M$, then the dimension of $M$ is odd or even according to whether $r$ is odd or even.

*Proof.* If $\lambda_k^2 - 1 < 0$, then the eigenvalues $\lambda_k \pm \sqrt{\lambda_k^2 - 1}$ are complex conjugate and they come in pairs. This implies that the sub-bundle $\Gamma_k$ is complex conjugate to the sub-bundle $\pi_k$. We also have the dimension of $\bar{\pi}_k$ is 1. This means that the dimension of $M_k$ is odd.

$$\text{Dimension of } M = \sum_{k=1}^{r} \text{dimension } M_k$$

and this is odd or even according to whether $r$ is odd or even. //

## EXAMPLES

1. If $\lambda_k = 0$, for all $k$, then

$$\text{(i)} \quad S^2 + I = u^k \otimes U_k$$
$$\text{(ii)} \quad SU_k = 0$$

and $M$ is an almost contact manifold (Mishra 1984, p. 211; Sasaki & Hatakeyama 1961).

2. Consider the Euclidean space $V_5$ on which

$$S=\begin{bmatrix}\cosh x & 0 & 0 & 0 & 0\\ 0 & \cosh x & \sinh x & 0 & 0\\ 0 & \sinh x & \cosh x & 0 & 0\\ 0 & 0 & 0 & \cosh x & \sinh x\\ 0 & 0 & 0 & \sinh x & \cosh x\end{bmatrix}$$

$$\begin{aligned}(S-\cosh xI)^2 &= \sinh^2 x\,\mathrm{diag}(0\quad 1\quad 1\quad 1\quad 1)\\ &= -(1-\cosh^2 x)\,\mathrm{diag}(0\quad 1\quad 1\quad 1\quad 1)\\ &= -(1-\cosh^2 x)\,\mathrm{diag}(1\quad 1\quad 1\quad 1\quad 1)\\ &\quad +(1-\cosh^2 x)\,\mathrm{diag}(1\quad 0\quad 0\quad 0\quad 0)\end{aligned}$$

$$\therefore\quad (S-\cosh xI)^2+(1-\cosh^2 x)I_5=(1-\cosh^2 x)[1\quad 0\quad 0\quad 0\quad 0]\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

or

$$H^2+KI_5=Ku\otimes U.$$

Hence $V_5$ is an almost contact $\lambda$-structure with $\lambda>1$.

3. Similar to Example 2, we can assume on $V_5$,

$$S=\begin{bmatrix}\cos\theta & 0 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0 & 0\\ 0 & \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 0 & \cos\theta & \sin\theta\\ 0 & 0 & 0 & \sin\theta & \cos\theta\end{bmatrix}.$$

If $H=(S-\cos\theta I)$, then

$$H^2+(1-\cos^2\theta)I_5=(1-\cos^2\theta)u\otimes U.$$

In this case $-1<\lambda<1$, and $V_5$ is an almost contact $\lambda$-manifold.

## REFERENCES

**Hoffman, K. & Kunze, R. 1963.** Linear algebra. Prentice Hall Inc.

**Mishra, R.S. 1984.** Structure of differentiable manifolds and their application. Chanrama Praka, Allahabad.

**Sasaki, S. & Hatakeyama, Y. 1961.** On differentiable manifolds with certain structures which are closely related to almost contact structures. Tohoku Mathematical Journal **13**: 281–94.

(*Received 4 June 1986, revised 23 November 1986*)

# منطويات $\lambda$ شبه التماسية

عدنان العقيل
قسم الرياضيات بجامعة الكويت ،
ص . ب ٥٩٦٩ ، الصفاة ١٣٠٦٠ ، الكويت

## خلاصة

نقدم في هذا البحث تعميما للمنطويات شبه التماسية . وعلى غرار حقـل الموتـرات $F$ على المنطويات شبه التماسية نقوم هنا بتعريف حقل موترات $S$ من النوع (١,١) ، ثم نستخدم هذا الحقل في منطويات $\lambda$ شبه التماسية .