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On the zeros of a polynomial and its derivative. II

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ABSTRACT

Let $p(z)$ be a polynomial of degree n having all its zeros in the closed unit disk. We show that if a is a zero of $p(z)$ of multiplicity k then for $k < n \leqslant (k+1)^2$ the disk $\left\{z: |z-a| \leqslant \frac{2k}{k+1}\right\}$ contains at least k zeros of the derivative $p'(z)$.

INTRODUCTION

We denote by $D(a; R)$ the open disk $\{z: |z-a| < R\}$ and by $\overline{D(a; R)}$ its closure. Let $p(z)$ be a polynomial of degree n having all its zeros in $\overline{D(0; 1)}$. Generalizing a result of Goodman, Rahman & Ratti (1969), Schmeisser (1969), Meir & Sharma (1969) we have recently proved (Tariq 1986) that if a is a zero of multiplicity $k(<n)$ of $p(z)$ then the disk $\overline{D\left(a; \frac{2k}{k+1}\right)}$ contains at least k zeros of $p'(z)$ provided $|a| = 1$. Here we show that the restriction $|a| = 1$ can be dropped for polynomials of degree n with $k < n \leqslant (k+1)^2$.

Theorem. Let $|a| \leqslant 1$. If

$$p(z) := c(z-a)^k \prod_{j=1}^{n-k} (z - z_j)$$

is a polynomial of degree n with $k < n \leqslant (k+1)^2$ such that $|z_j| \leqslant 1$ for $j = 1, \ldots, n-k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $\overline{D\left(a; \frac{2k}{k+1}\right)}$.

PROOF OF THE THEOREM

As mentioned, above, it was proved in Tariq (1986) that the disk $\overline{D\left(a; \frac{2k}{k+1}\right)}$. contains at least k zeros of $p'(z)$ in the case $|a| = 1$ if only $n > k$. So we may assume

$0 \leqslant |a| < 1$. Besides, there is no loss of generality in supposing that $0 \leqslant a < 1$. Then clearly the polynomial

$$P(z) := (az - 1)^n \, p\left(\frac{z - a}{az - 1}\right)$$

has a zero of multiplicity k at the origin and all its other zeros lie in $\overline{D(0; 1)}$. Hence it has the form

$$P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_k z^k = z^k (c_n z^{n-k} + c_{n-1} z^{n-k-1} + \cdots + c_k)$$

where

$$\left|\frac{c_k}{c_n}\right| \leqslant 1 \quad \text{and} \quad \left|\frac{c_{n-1}}{c_n}\right| \leqslant (n - k) \quad .$$

Since

$$P'(z) = na(az - 1)^{n-1} p\left(\frac{z - a}{az - 1}\right) - (1 - a^2)(az - 1)^{n-2} p'\left(\frac{z - a}{az - 1}\right)$$

we have

$$\begin{aligned}(1 - a^2)(az - 1)^{n-1} p'\left(\frac{z - a}{az - 1}\right) &= naP(z) - (az - 1)P'(z) \\ &= (nc_n + ac_{n-1}) z^{n-1} + \cdots + kc_k z^{k-1} \quad .\end{aligned}$$

Consequently, the zeros of $p'\left(\frac{z - a}{az - 1}\right)$ are the same as those of the polynomial

$$z^{k-1}\left(z^{n-k} + \cdots + \frac{kc_k}{nc_n + ac_{n-1}}\right) \quad ,$$

i.e. it has a $(k - 1)$-fold zero at the origin and at least one zero in

$$\left\{z\colon\ 0 < |z| \leqslant \left(\frac{k}{n - (n - k)a}\right)^{1/n-k)}\right\}.$$

Hence $p'\left(\frac{z - a}{az - 1}\right)$ has at least k zeros in

$$\overline{D\left(0; \left(\frac{k}{n - (n - k)a}\right)^{1/(n-k)}\right)}.$$

Setting

$$\eta := \left(\frac{k}{n - (n - k)a}\right)^{1/(n-k)}$$

we readily deduce that $p'(z)$ has at least k zeros in $\overline{D\left(\frac{a(1 - \eta^2)}{1 - a^2\eta^2}; \frac{\eta(1 - a^2)}{1 - a^2\eta^2}\right)}$. It is now enough to show that

$$\overline{D\left(\frac{a(1 - \eta^2)}{1 - a^2\eta^2}; \frac{\eta(1 - a^2)}{1 - a^2\eta^2}\right)} \subseteq \overline{D\left(a; \frac{2k}{k + 1}\right)} \quad . \tag{1}$$

Since the disk on the left-hand side of (1) has the real interval $\left[\frac{a-\eta}{1-a\eta}, \frac{a+\eta}{1+a\eta}\right]$ as a diameter, it is sufficient to check that

$$a - \frac{a-\eta}{1-a\eta} \leqslant \frac{2k}{k+1}$$

which is equivalent to

$$(1-a^2)\frac{k+1}{2k} + a \leqslant \left(\frac{n-(n-k)a}{k}\right)^{1/(n-k)} . \qquad (2)$$

Setting $n = \lambda k$ in (2) we see that the theorem will be proved if we show that the inequality

$$f(k, a, \lambda) := (\lambda - 1)a + \left\{a + \frac{k+1}{2k}(1-a^2)\right\}^{(\lambda-1)k} \leqslant \lambda \qquad (3)$$

holds for $0 \leqslant a \leqslant 1$ and $1 < \lambda \leqslant \frac{(k+1)^2}{k}$. Since f is a continuous function of a and assumes the value λ for $a = 1$, it is more than enough to demonstrate that it increases with a on the interval $[0, 1)$. Calculating the partial derivative of f with respect to a we obtain

$$\frac{1}{\lambda - 1}\frac{\partial}{\partial a} f(k, a, \lambda) = 1 + g(k, a, \lambda)$$

where

$$g(k, a, \lambda) := \{k - (k+1)a\}\left\{a + \frac{k+1}{2k}(1-a^2)\right\}^{(\lambda-1)k-1} .$$

Since $g(k, a, \lambda)$ is positive for $0 \leqslant a < \frac{k}{k+1}$ the same can be said about $\frac{\partial}{\partial a} f(k, a, \lambda)$.

Now we show that $g(k, a, \lambda)$ decreases from $g\left(k, \frac{k}{k+1}, \lambda\right) = 0$ to $g(k, 1, \lambda) = -1$ as a increases from $\frac{k}{k+1}$ to 1. For this we have to look at the sign of $\frac{\partial}{\partial a} g(k, a, \lambda)$ for $a \in \left[\frac{k}{k+1}, 1\right)$. A simple calculation shows that

$$\frac{\partial}{\partial a} g(k, a, \lambda) < 0$$

if and only if

$$\{(\lambda - 1)k - 1\}\{k - (k+1)a\}^2 < \frac{k+1}{2}\{2ak + (k+1)(1-a^2)\} . \qquad (4)$$

We will like this to be true for $1 < \lambda \leqslant \frac{(k+1)^2}{k}$. Since the left-hand side of (4) is an increasing function of λ whereas the right-hand side does not depend on it, we need to verify the inequality only for $\lambda = \frac{(k+1)^2}{k}$. In that case it reduces to

$$(k^2 + k)\{k - (k+1)a\}^2 < \frac{k+1}{2}\{2ak + (k+1)(1-a^2)\}$$

which is equivalent to

$$\left(a - \frac{k-1}{k+1}\right)(a-1) < 0 \quad . \tag{5}$$

This latter inequality is obviously true for $a \in \left(\frac{k-1}{k+1}, 1\right)$ and so *a fortiori* for $a \in \left[\frac{k}{k+1}, 1\right)$. Thus (4) holds for $\lambda = \frac{(k+1)^2}{k}$ and consequently for $\lambda < \frac{(k+1)^2}{k}$ as well. This implies that

$$g(k, a, \lambda) > g(k, 1, \lambda) = -1 \quad \text{for } \frac{k}{k+1} \leqslant a < 1 \quad .$$

Hence $\frac{\partial}{\partial a} f(k, a, \lambda) = (\lambda - 1)\{1 + g(k, a, \lambda)\}$ is positive not only for $0 \leqslant a < \frac{k}{k+1}$ but also for $\frac{k}{k+1} \leqslant a < 1$, i.e. $f(k, a, \lambda)$ increases from $f(k, 0, \lambda)$ to $f(k, 1, \lambda) = \lambda$ as a increases from 0 to 1. With this the proof of inequality (3) is complete and so is the proof of our theorem.

A COROLLARY

Our theorem implies in particular that if $|a| \leqslant 1$ and $p(z) := c(z-a)^k \prod_{j=1}^{n-k} (z - z_j)$ is a polynomial of degree n with $2 \leqslant k < n \leqslant k + 4$ and $|z_j| \leqslant 1$ for $j = 1, \ldots, n-k$, then $p'(z)$ has at least k zeros in $\overline{D\left(a, \frac{2k}{k+1}\right)}$. Since this is also known to be true for $k = 1$ (Meir & Sharma 1969) we may state the following:

Corollary. Let $|a| \leqslant 1$. If $p(z) := c(z-a)^k \prod_{j=1}^{n-k} (z - z_j)$ is a polynomial of degree n with $k < n \leqslant k + 4$ such that $|z_j| \leqslant 1$ for $j = 1, \ldots, n - k$, then taking multiplicity into account, $p'(z)$ has at least k zeros in $\overline{D\left(a; \frac{2k}{k+1}\right)}$.

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حول أصفار الحدودية ومشتقتها (الجزء الثاني)

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خلاصة

لنفرض أن $p(z)$ حدودية من الدرجة n تأخذ جميع أصفارها في قرص الوحدة المغلق . في هذا البحث نثبت أنه إذا كانت a صفرا للحدودية $p(z)$ ذا تضاعف k ، فللقيم $k<n\leqslant(k+1)^2$ ، يحتوي القرص $\left\{z: |z - a| \leqslant \frac{2k}{k + 1}\right\}$ على k من أصفار المشتقة $p'(z)$ على الأقل .