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On the spaces of convolution operators and multipliers in $\mathcal{K}'_M$

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ABSTRACT

Let $\mathcal{K}'_M$ be the space of distributions which grow not faster than $\exp(M(kx))$, for some nonnegative integer k and an index function M. We study several topologies on $O'_c(\mathcal{K}'_M:\mathcal{K}'_M)$, the space of convolution operators in $\mathcal{K}'_M$, and $O_c(\mathcal{K}'_M:\mathcal{K}'_M)$, its dual. We also consider the space $O_m(\mathcal{K}'_M:\mathcal{K}'_M)$ of multipliers of $\mathcal{K}'_M$ and provide it with several topologies.

INTRODUCTION

Convolution operators and multipliers of the spaces $\mathcal{K}'_p$; $p \geqslant 1$, of distributions of exponential growth were characterized by Hasumi (1961) in the case $p = 1$, and Sampson & Zielezny (1976) in the case $p > 1$. Zielezny (1968) provided the space $O'_c(\mathcal{K}'_1:\mathcal{K}'_1)$ of convolution operators in $\mathcal{K}'_1$ with several topologies, which turned out to be equivalent. Also, he has shown that the space $O'_c(\mathcal{K}'_1:\mathcal{K}'_1)$ is complete, nuclear, bornological and Montel. In this paper we study the space $O'_c(\mathcal{K}'_M:\mathcal{K}'_M)$ of convolution operators in $\mathcal{K}'_M$, where $\mathcal{K}'_M$ is the space of distributions of rapid growth. We provide this space with several topologies and prove relations among these topologies. Provided with some of the topologies, the space $O'_c(\mathcal{K}'_M:\mathcal{K}'_M)$ is complete and nuclear. We also consider the space $O_c(\mathcal{K}'_M:\mathcal{K}'_M)$, the dual of $O'_c(\mathcal{K}'_M:\mathcal{K}'_M)$. Finally we study the space $O_m(\mathcal{K}'_M:\mathcal{K}'_M)$ of multipliers of $\mathcal{K}'_M$. We characterize this space and provide it with several topologies, and prove that all the topologies on the space are equivalent. The space $O'_c(\mathcal{K}'_M:\mathcal{K}'_M)$ was characterized by Pahk (1981); his result is an immediate generalization of a similar characterization of $O'_c(\mathcal{K}'_p:\mathcal{K}'_p)$, $p > 1$, due to Sampson & Zielezny (1976).

By N^n, R^n we denote the sets of n-tuples of nonnegative integers and real numbers respectively. For $\alpha = (\alpha_1, \dots, \alpha_n)$ in N^n we denote by $|\alpha|$ the sum $\alpha_1 + \cdots + \alpha_n$. By $\mathcal{D}$ and $\mathcal{D}'$ we denote Schwartz spaces of test functions and distributions, by $\mathcal{S}$ we denote the space of infinitely differentiable functions rapidly decreasing at infinity, and its strong dual $\mathcal{S}'$ is the space of tempered distribution (see Schwartz (1965) for definitions of these spaces and their topologies). For any distribution T we denote by $\check{T}$ its symmetry with respect to the origin and by $\tau_h T$, $h \in R^n$, the translation of T by h. For $\alpha \in N^n$ we denote by D^α the differential operator $D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$, where

$D_j = (1/i)(\partial/\partial x_j)$, $j = 1, 2, \ldots, n$. Let E be a locally convex topological vector space and E' its strong dual. For a bounded subset B of E we denote by B^0 the polar of B, which is the set of all T in E' such that $|\langle T, \phi\rangle| < 1$ for all ϕ in B.

THE SPACES $\mathscr{K}_M$ AND $\mathscr{K}'_M$

Let $\mu(\xi)$; $0 \leqslant \xi \leqslant \infty$ be a nonnegative increasing function with $\mu(0) = 0$, $\mu(\infty) = \infty$. For any x, $0 \leqslant x < \infty$ define the function $M(x) = \int_0^x \mu(\xi)\, d\xi$. It follows that M is positive, increasing and convex. For negative x we define M by symmetry, i.e. $M(x) = M(-x)$. For $x = (x_1, x_2, \ldots, x_n)$ we define $M(x) = M(x_1) + M(x_2) + \cdots + M(x_n)$. We write w^k and w^{-k} for the functions $\exp[M(kx)]$ and $\exp[-M(kx)]$ respectively, $k = 0, 1, 2, \ldots$. By $w^{-k}\mathscr{S}$, $k = 0, 1, 2, \ldots$ we denote the space of all $w^{-k}\phi$; $\phi \in \mathscr{S}$, provided with the topology reflected on it by $\mathscr{S}$, i.e. a sequence $(\psi_j = w^{-k}\phi_j)$ converges in $w^{-k}\mathscr{S}$ if and only if the sequence (ϕ_j) converges in $\mathscr{S}$. Hence, the space $w^{-k}\mathscr{S}$ has the same topological properties of $\mathscr{S}$, and for $k > l$ the inclusion $w^{-k}\mathscr{S} \hookrightarrow w^{-l}\mathscr{S}$ is continuous. Without loss of generality we will assume that the function $\mu(\xi)$ is smooth. Now we give the definition of the test function space $\mathscr{K}_M$.

By $\mathscr{K}_M$ we denote the intersection of the spaces $w^{-k}\mathscr{S}$ for all $k = 0, 1, 2, 3, \ldots$, provided with the projective limit topology of the spaces $w^{-k}\mathscr{S}$. The intersection is nontrivial since it contains the space of all infinitely differentiable functions with compact support. Since the projective limit is the coarsest topology on $\mathscr{K}_M$ so that each of the embeddings $u_k: \mathscr{K}_M \to w^{-k}\mathscr{S}$ is continuous, and since $\mathscr{S}$ is locally convex Hausdorff topological vector space, it follows that $\mathscr{K}_M$ is a locally convex Hausdorff topological vector space (Theorem 5.1, Chapter 2 of Schaefer (1980)).

We remark that it follows immediately from the definition of $\mathscr{K}_M$ that another equivalent topology is the one defined by the family of semi-norms

$$v_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leqslant k}} (1 + |x|^2)^k |D^\alpha(w^k\phi)(x)|; \quad k = 0, 1, 2, \ldots.$$

Moreover, the topology of $\mathscr{K}_M$ generated by the family $\{v_k\colon k = 0, 1, 2, \ldots\}$ is equivalent to the topology defined by the family of semi-norms

$$\omega_k(\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leqslant k}} w^k |D^\alpha\phi(x)|; \quad k = 0, 1, 2, \ldots.$$

It has been proved in Abdullah (1987) that $\mathscr{K}_M$ is a Frechet nuclear space. Moreover, it is barreled, completely reflexive, bornologic and is a normal space of distributions. By $\mathscr{K}'_M$ we denote the strong dual of $\mathscr{K}_M$ provided with the topology of uniform convergence on bounded subsets of $\mathscr{K}_M$; $\mathscr{K}'_M$ is the space of distributions which grow not faster than $\exp(M(kx))$ for some $k \geqslant 0$. The elements of $\mathscr{K}'_M$ are called distributions of rapid growth. $\mathscr{K}'_M$ is bornological because it is the strong dual of a reflexive Frechet space. For $T \in \mathscr{K}'_M$ and $\phi \in \mathscr{K}_M$ we define the convolution of T and ϕ by the relation $(T * \phi)(x) = \langle T_y, \phi(x - y)\rangle$. Now we give a theorem which characterizes the elements of $\mathscr{K}'_M$.

Theorem A (Pahk (1981), Chapter 1, Theorem 1.2.3). Let T be any distribution, the following statements are equivalent.

(i) T is in $\mathscr{K}'_M$.
(ii) $T = D^\alpha[w^k f]$, for some multi-index α, a positive integer k and a bounded continuous function f.
(iii) There exists a positive integer k_1 so that $(T * \phi)(x) = O(w^{k_1})$, for all ϕ in $\mathscr{D}$.

THE SPACE OF CONVOLUTION OPERATORS IN $\mathscr{K}'_M$ AND ITS DUAL

We define the space $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$ as the intersection $\bigcap_{k=0}^{\infty} w^{-k}\mathscr{S}'$ provided with the projective limit topology of the spaces $w^{-k}\mathscr{S}'$, as $k \to \infty$. The following theorem characterizes the elements of $\mathscr{K}'_M$ which are in $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$.

Theorem 1. Let S be any element of $\mathscr{K}'_M$. The following statements are equivalent.

(1) S is in $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$.
(2) The distributions $w^k S$, $k = 0, 1, 2, \ldots$, are in $\mathscr{S}'$.
(3) For any $k \geqslant 0$ there exists a nonnegative integer m such that

$$S = \sum_{|\alpha| \leqslant m} D^\alpha f_\alpha,$$

where, for each α, f_α is a continuous function such that $w^k f_\alpha \in L^\infty$.
(4) For any k, the set of distributions $\{w^{-k}(h)\tau_h S : h \in R^n\}$ is bounded in $\mathscr{D}'$.
(5) $S * \phi$ is in $\mathscr{K}_M$ for all ϕ in $\mathscr{K}_M$ and the map $\phi \to S * \phi$ from $\mathscr{K}_M$ into $\mathscr{K}_M$ is continuous.

The equivalence of (1) and (2) is obvious, the proof of the equivalence of the statements (2) to (5) is similar to that of Theorem 2 of Sampson & Zielezny (1976) and will be omitted.

For $S \in O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$ and $T \in \mathscr{K}'_M$ we define $S * T$, the convolution of S and T, by $\langle S * T, \phi \rangle = \langle T, \check{S} * \phi \rangle$; $\phi \in \mathscr{K}_M$, where $(\check{S} * \phi)(x) = \langle S_y, \check{\phi}(x - y) \rangle$. Let (T_j) be a sequence in $\mathscr{K}'_M$ converging to 0. From property (5) of the above theorem, it follows that $\check{S} * B$ is bounded in $\mathscr{K}_M$ for every bounded subset B of $\mathscr{K}_M$. Hence $\langle S * T_j, \phi \rangle = \langle T_j, \check{S} * \phi \rangle \to 0$ uniformly in $\phi \in B$. Since $\mathscr{K}'_M$ is bornological it follows that the map $T \to S * T$ from $\mathscr{K}'_M$ into $\mathscr{K}'_M$ is continuous. The space $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$ is the space of convolution operators in $\mathscr{K}'_M$.

The space $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$ will be provided with several topologies. The first topology τ_p is the projective limit topology of the spaces $w^{-k}\mathscr{S}'$. Statement (5) of Theorem 1 suggests a second topology τ_b induced by $L_b(\mathscr{K}_M : \mathscr{K}_M)$, the space of all continuous linear maps from $\mathscr{K}_M$ into itself provided with the topology of uniform convergence on bounded subsets of $\mathscr{K}_M$. A third topology τ'_b is the one induced by $L_b(\mathscr{K}'_M : \mathscr{K}'_M)$, the space of all continuous linear maps from $\mathscr{K}'_M$ into itself provided with the topology of uniform convergence on bounded subsets of $\mathscr{K}'_M$. We have the following:

Theorem 2. On $O'_c(\mathscr{K}'_M : \mathscr{K}'_M)$ the topologies τ_b and τ'_b are equal.

Proof. We will prove that τ_b is finer than τ'_b. The proof that τ'_b is finer than τ_b is similar and will be omitted. Let U be a member of 0-neighborhood base for τ'_b.

Without loss of generality we can assume that

$$U = U(V', B') = \{S \in O_c'(\mathscr{K}'_M : \mathscr{K}'_M) \colon S * T \in V' \text{ for all } T \in B'\},$$

where B' is a bounded subset of $\mathscr{K}'_M$ and V' is a neighborhood of 0 in $\mathscr{K}_M$. We can assume that

$$V' = V'(B, \varepsilon) = \{T \in \mathscr{K}'_M \colon |\langle T, \phi \rangle| < \varepsilon \text{ for all } \phi \in B\},$$

where B is a bounded subset of $\mathscr{K}_M$ and ε is a positive number. Let

$$V = \{\phi \in \mathscr{K}_M \colon |\langle T, \phi \rangle| < \varepsilon \text{ for all } T \in B'\},$$

since $\mathscr{K}_M$ is barreled it follows that V is a neighborhood of 0 in $\mathscr{K}_M$. Without loss of generality we will assume that $B = \check{B} = \{\check{\phi} \colon \phi \in B\}$ and $B' = \check{B}' = \{\check{T} \colon T \in B'\}$. Let $W = W(V, B) = \{S \in O_c'(\mathscr{K}'_M : \mathscr{K}'_M) \colon S * \phi \in V \text{ for all } \phi \text{ in } B\}$, where V and B are as above. We claim that $W(V, B)$ is contained in $U(V', B')$. For, let $S \in W(V, B)$, then for any $T \in B'$ and any $\phi \in B$, one has $|\langle S * T, \phi \rangle| = |\langle T, \check{S} * \phi \rangle| < \varepsilon$, i.e. $S * T \in V'$ for all $T \in B'$. This completes the proof of the theorem.

The space $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$ with τ_b is complete and nuclear. Indeed, since $\mathscr{K}_M$ is bornological and complete it follows that $L_b(\mathscr{K}_M : \mathscr{K}_M)$ is complete and nuclear. Being a closed subspace of $L_b(\mathscr{K}_M : \mathscr{K}_M)$, it follows that $(O_c'(\mathscr{K}'_M : \mathscr{K}'_M), \tau_b)$ is complete and nuclear. Moreover, since $(O_c'(\mathscr{K}'_M : \mathscr{K}'_M), \tau_p)$ is the projective limit of the nuclear spaces $w^{-k}\mathscr{S}'$, it follows from the corollary to Theorem 7.4, Chapter 3 of Schaefer (1980) that $(O_c'(\mathscr{K}'_M : \mathscr{K}'_M), \tau_p)$ is nuclear.

We denote by $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$ the union of the spaces $w^k\mathscr{S}$, $k = 0, 1, 2, \ldots$, provided with the inductive limit topology. Since $\mathscr{S}$ is bornological (being metrizable), and the inductive limit of bornological spaces is bornological, it follows that $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$ is bornological. The following theorem gives the relation between $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$ and $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$.

Theorem 3. The space $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$ is the strong dual of $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$ with the topology τ_p.

Proof. Since $\mathscr{S}'$ is Montel, it is barreled, hence it is a Mackey space. Thus $w^{-k}\mathscr{S}'$ is a Mackey space. By Theorem 4.4, Chapter 4 of Schaefer (1980) it follows that

$$[O_c'(\mathscr{K}'_M : \mathscr{K}'_M)]' = \left[\operatorname*{Proj}_{k \to \infty} w^{-k}\mathscr{S}'\right]' = \operatorname*{Induc.}_{k \to \infty} w^k\mathscr{S} = O_c(\mathscr{K}'_M : \mathscr{K}'_M).$$

Theorem 3 suggests a fourth topology on $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$, namely the strong dual topology, which is the topology of uniform convergence on bounded subsets of $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$. Let B_1 be a bounded subset of $O_c(\mathscr{K}'_M : \mathscr{K}'_M)$. We would like to know what the elements of B_1^0 look like. By definition, there exists a $k \geqslant 0$ such that B_1 is bounded in w^k similar to $\mathscr{S}$, hence $B_1 = w^k B$ for some bounded subset B of $\mathscr{S}$. Thus $B_1^0 = (w^k B)^0$. Let $T \in B_1^0$, i.e. $|\langle T, \phi \rangle| < 1$ for all ϕ in B_1, or equivalently $|\langle T, w^k\psi \rangle| < 1$ for all ψ in B. This implies that $w^k T \in B^0$, i.e. $T \in w^{-k}B^0$. The converse is equally true, i.e. any element of $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$ which belongs to $w^{-k}B^0$ is in the polar of B_1. Thus $B_1^0 = w^{-k}B^0 \cap O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$. We are in a position to prove the following:

Theorem 4. On $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$ the strong dual topology coincides with the projective limit topology τ_p.

Proof. Let V be a member of a 0-neighborhood base in τ_p. Without loss of generality one can assume that $V = w^{-k}B^0 \cap O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$ for some $k \geqslant 0$ and a bounded subset B of $\mathscr{S}$. From the discussion that precedes the theorem, it follows that $V = B_1^0$ for some B_1, a bounded subset of O_c; hence V is a member of 0-neighborhood base for the strong dual topology. This shows that the strong dual topology is finer than the projective limit topology. The proof that τ_p is not weaker than the strong dual topology is similar to the above and will be omitted.

As a result of Theorem 3, $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$ is strong dual of $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$. We can provide $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$ with the strong dual topology, i.e. the topology of uniform convergence on bounded subsets of $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$. We have the following:

Theorem 5. On $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$ the inductive limit topology and the strong dual topology coincide.

Proof. First we prove that the inductive limit topology is finer than the strong dual topology. For this it suffices to show that the identity map from $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$ with the inductive limit topology to itself with the strong dual topology is continuous. Since $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$ with the inductive limit topology is bornological, it suffices to show that the map is sequentially continuous (Theorem 8.3, Chapter 2 of Schaefer (1980)). Let (f_j) be a sequence converging to 0 in the inductive limit topology. By definition, there exists $k \geqslant 0$ such that $w^{-k}f_j \to 0$ in $\mathscr{S}$. Let B' be any bounded subset of $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$. Since $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$ is the projective limit of $w^{-k}\mathscr{S}'$, it follows that w^kB' is a bounded subset of $\mathscr{S}'$. Since $\mathscr{S}$ is reflexive it follows that

$$\langle f_j, T\rangle = \langle w^{-k}f_j, w^kT\rangle \to \quad \text{uniformly in } T \in B',$$

i.e. (f_j) converges to 0 uniformly on bounded subsets of $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$. This establishes continuity of the identity.

Secondly, we prove that the strong dual topology is not weaker than the inductive limit topology. Let W be a neighborhood of 0 in the inductive limit topology. We find V a member of 0-neighborhood base in the strong dual topology such that $V \subset W$. Without loss of generality we can assume that W is closed and convex. Let $W_k = W \cap w^k\mathscr{S}$. W_k is a neighborhood of 0 in $w^k\mathscr{S}$ for all $k \geqslant 0$, i.e. $w^{-k}W_k$ is a neighborhood of 0 in $\mathscr{S}$. Hence there exists B'_k, a bounded subset of $\mathscr{S}'$, such that $w^{-k}W_k = (B'_k)^0$, i.e. $W_k = w^k(B'_k)^0$. For each $k = 0, 1, 2, \ldots$, let $\bar{B}_k$ be the smallest closed, convex and bounded subset of $\mathscr{S}'$ containing $B'_k \cup \{0\}$. The set $B = \bigcap_{k=0}^{\infty} w^{-k}\bar{B}_k$ is a bounded subset of $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$ with the topology τ_p. Indeed, let U be a member of 0-neighborhood base for $(O'_c(\mathscr{K}'_M:\mathscr{K}'_M), \tau_p)$; hence U is the intersection of $w^{-k}B_k^0$, where B_k is a bounded subset of $\mathscr{S}$, and k belongs to a finite index set. Hence there exists an F, a bounded subset of $\mathscr{S}$, such that $U = w^{-s}F^0$ for some positive integer s. Since $\bar{B}_s$ is bounded in $\mathscr{S}'$, there exists $\lambda > 0$ such that $\lambda\bar{B}_s \subset F^0$; hence

$$\lambda B \subset \lambda w^{-s}\bar{B}_s \subset w^{-s}F^0 = U.$$

Let $V = B^0$ the polar of B, then V is a member of 0-neighborhood base for the strong dual topology of $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$. Moreover, from Corollary 2 to Theorem 1.5 of Chapter

4 of Schaefer (1980) one has

$$V = B^0 = \left(\bigcap_{k=0}^{\infty} w^{-k}\bar{B}_k\right)^0 = \Gamma\left(\bigcup_{k=0}^{\infty} (w^{-k}\bar{B}_k)^0\right);$$

$$= \Gamma\left(\bigcup_{k=0}^{\infty} w^k(\bar{B}_k)^0\right)$$

$$\subset \Gamma\left(\bigcup_{k=0}^{\infty} w^k(B'_k)^0\right) = \Gamma\left(\bigcup_{k=0}^{\infty} W_k\right) \subset W,$$

where Γ denotes the closed convex hull. This completes the proof of the theorem.

Remark. The author could not prove that the topologies τ_b and τ_p on $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$ are equivalent.

THE SPACE OF MULTIPLIERS OF $\mathscr{K}'_M$

We define the space $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ to be the space of all infinitely differentiable functions f such that for every multi-index α there exists a positive integer k such that $D^\alpha f(x) = O(w^k)$ as $|x| \to \infty$. From the definition of $\mathscr{K}_M$ and Leibniz formula it follows that $f\phi \in \mathscr{K}_M$ whenever $\phi \in \mathscr{K}_M$. We provide $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ with the topology τ generated by the semi-norms

$$\rho_{k,\phi}(f) = \sup_{\substack{x \in R^n \\ |a| \leqslant k}} w^k |D^\alpha(f\phi)(x)|; \quad k = 0, 1, 2, \ldots, \quad \phi \in k_M.$$

The following theorem characterizes the elements of $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$. The theorem appears in Abdullah (1987) and we present the proof here for the sake of completeness.

T heorem 6. Let $f \in C^\infty(R^n)$. The following statements are equivalent.

(1) f is in $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$.
(2) The linear mapping $\phi \to f\phi$ from $\mathscr{K}_M$ into itself is continuous.
(3) The linear mapping $T \to fT$ from $\mathscr{K}'_M$ into itself is continuous.

Proof. The implication (1) ⇒ (2). Let k be any nonnegative integer, $\alpha \in N^n$ with $|\alpha| \leqslant k$. By Leibniz formula and the definition of $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ it follows that for any ϕ in $\mathscr{K}_M$ one has

$$|D^\alpha(f\phi)| \leqslant \sum_{\beta \leqslant \alpha} c_1 c_2 w^r |D^{\alpha-\beta}\phi|, \quad \beta \in N^n,$$

where $c_1 = c_1(\alpha, \beta)$, $c_2 = c_2(\beta, f)$ and $r = r(\beta, f)$ are constants. Hence

$$\sup_{\substack{x \in R^n \\ |a| \leqslant k}} w^k |D^\alpha(f\phi)| \leqslant c \sup_{\substack{x \in R^n \\ |\gamma| \leqslant k_1}} w^{k_1} |D^\gamma \phi|,$$

for some constant c and $k_1 = k + r$. This inequality implies that $f\phi$ is in $\mathscr{K}_M$ and the continuity of the linear map $\phi \to f\phi$.

The implication (2) ⇒ (3). For $f \in O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ we define fT by $\langle fT, \phi\rangle = \langle T, f\phi\rangle$, $\phi \in \mathscr{K}_M$. Hence fT is well defined as a linear functional on $\mathscr{K}_M$. To show the continuity of fT as a map from $\mathscr{K}_M$ to $\mathbb{C}$, it suffices to show that it is sequentially continuous because $\mathscr{K}_M$ is bornological. Let (ϕ_j) be a sequence in $\mathscr{K}_M$ converging to 0. By (2) the sequence $(f\phi_j)$ converges to 0 in $\mathscr{K}_M$, hence $\langle fT, \phi_j\rangle = \langle T, f\phi_j\rangle \to 0$ by the continuity of T. Thus $fT \in \mathscr{K}'_M$. Since $\mathscr{K}'_M$ is bornological, to show the continuity of the map $T \to fT$ it suffices to show that it is sequentially continuous, which follows immediately from (2).

The implication (3)$\Rightarrow$(2). For every $\phi \in \mathscr{K}_M$ define on $\mathscr{K}'_M$ the linear form Λ_ϕ: $T \to \langle fT, \phi\rangle = \langle T, f\phi\rangle$. The continuity of the map $T \to fT$ from $\mathscr{K}'_M$ into itself implies that Λ_ϕ is continuous, hence $\Lambda_\phi \in \mathscr{K}''_M$, the double dual of $\mathscr{K}_M$. Since $\mathscr{K}_M$ is reflexive, there exists $\psi \in \mathscr{K}_M$ so that $\Lambda_\phi(T) = \langle fT, \phi\rangle = \langle T, f\phi\rangle = \langle T, \psi\rangle$ for all T in $\mathscr{K}'_M$, i.e. $f\phi = \psi \in \mathscr{K}_M$ for all ϕ in $\mathscr{K}_M$. Since $\mathscr{K}_M$ is metrizable and reflexive, and $fT \in \mathscr{K}'_M$ for all T in $\mathscr{K}'_M$, it follows that the map $\phi \to f\phi$ from $\mathscr{K}_M$ into itself is continuous.

The implication (2)$\Rightarrow$(1). If f is not in $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$, then there exists a sequence $(x_j) \subset R^n$, $|x_{j+1}| > j + |x_j|$ such that $|f(x_j)| > j^2 w^j(x_j)$. Let $\chi \in \mathscr{D}(\overline{B(0,1)})$, $\chi(x) = 1$ if $|x| \leqslant \frac{1}{2}$. Let $\phi_j(x) = (1/j)w^{-j}(x_j)\chi(x - x_j)$; $j = 1, 2, \ldots$. Then $\phi_j \to 0$ in $\mathscr{K}_M$, but $(f\phi_j)$ does not converge to 0 in $\mathscr{K}_M$ since $|f(x_j)\phi_j(x_j)| > j$ for all j. The contradiction proves the implication.

The space $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ is the space of multipliers of $\mathscr{K}'_M$. From the above theorem it follows that one can provide $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ with two more topologies. The first one, τ_b, is the topology induced on it by $L_b(\mathscr{K}_M:\mathscr{K}_M)$, the space of all continuous linear maps from $\mathscr{K}_M$ into itself provided with the topology of uniform convergence on bounded subsets of $\mathscr{K}_M$. The second one τ'_b is the topology induced on it by $L_b(\mathscr{K}'_M:\mathscr{K}'_M)$, the space of all continuous linear maps from $\mathscr{K}'_M$ into itself provided with the topology of uniform convergence on bounded subsets of $\mathscr{K}'_M$. Moreover, $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ could be provided with τ_s, the topology of simple (pointwise) convergence on $\mathscr{K}_M$. The relation between the topologies on $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ is given by the following:

Theorem 7. On $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$, the topologies τ, τ_s, τ_b and τ'_b are equivalent.

Proof. (1) τ is equivalent to τ_s. Let W be a member of 0-neighborhood base in τ. There exist $\psi \in \mathscr{K}_M$, k nonnegative integer and $\varepsilon > 0$ such that

$$W = \{f \in O_m(\mathscr{K}'_M:\mathscr{K}'_M)\colon \rho_{k,\psi}(f) < \varepsilon\} = \{f \in O_m(\mathscr{K}'_M:\mathscr{K}'_M)\colon \rho_k(f\psi) < \varepsilon\} = N(V, \psi),$$

where $V = \{\phi \in \mathscr{K}_M\colon \rho_k(\phi) < \varepsilon\}$ is a neighborhood of 0 in $\mathscr{K}_M$. Since $N(V, \psi)$ is a member of 0-neighborhood base in τ_s, the assertion follows.

(2) τ_b is equivalent to τ'_b. Let $N(V, B) = \{f \in O_m(\mathscr{K}'_M:\mathscr{K}'_M)\colon f\phi \in V \text{ for all } \phi \text{ in } B\}$ (where V is a neighborhood of 0 in $\mathscr{K}_M$ and B is a bounded subset of $\mathscr{K}_M$) be a member of 0-neighborhood base in τ_b. Since $\mathscr{K}_M$ is reflexive, we can assume that V is the polar $(B')^0$ of a bounded subset B' of $\mathscr{K}'_M$. Let $V' = B^0$, V' is a member of 0-neighborhood base in $\mathscr{K}'_M$. Moreover,

$$N'(V', B') = \{g \in O_m(\mathscr{K}'_M:\mathscr{K}'_M)\colon gT \in V' \text{ for all } T \in B'\}$$

is a member of 0-neighborhood base in τ'_b. We claim that $N'(V', B')$ is contained in $N(V, B)$. For, let $g \in N'(V', B')$, we show that $g\phi \in V$ for all ϕ in B. Since $gT \in V' = B^0$ one has

$$|\langle g\phi, T\rangle| = |\langle \phi, gT\rangle| < 1 \quad \text{for all } T \text{ in } B', \qquad \text{i.e. } g \in N(V, B).$$

Conversely, let $N'(V', B')$ be a member of 0-neighborhood base in τ_b', i.e. $N'(V', B') = \{g \in O_m(\mathscr{K}'_M : \mathscr{K}'_M) : gT' \in V'$ for all T in $B'\}$, where V' is a neighborhood of 0 in $\mathscr{K}'_M$ and B' is a bounded subset of $\mathscr{K}'_M$. Without loss of generality we can assume that V' is B^0, the polar of some bounded subset B of $\mathscr{K}_M$. Let $V = (B')^0$, then V is a neighborhood of 0 in $\mathscr{K}_M$. We claim that $N(V, B) = \{f \in O_m(\mathscr{K}'_M : \mathscr{K}'_M)$: $f\phi \in V$ for all ϕ in $B\}$ is contained in $N'(V', B')$. Indeed, let $f \in N(V, B)$, i.e. $f\phi \in V$ for all ϕ in B. Hence $|\langle fT, \phi\rangle| = |\langle f\phi, T\rangle| < 1$ for all T in B' and ϕ in B. Thus $fT \in B^0 = V'$ for all T in B', i.e. $f \in N'(V', B')$.

(3) τ_s is equivalent to τ_b. We show first that τ_s is not weaker than τ_b. Let $N(V, B) = \{f \in O_m(\mathscr{K}'_M : \mathscr{K}'_M)$: $f\phi \in V$ for all ϕ in $B\}$ (where V is a neighborhood of 0 in $\mathscr{K}_M$ and B a bounded subset of $\mathscr{K}_M$) be a member of 0-neighborhood base in τ_b. Without loss of generality we can assume that

$$V = \left\{g \in \mathscr{K}_M : \rho_k(g) = \sup_{\substack{x \in R^n \\ |\alpha| \leqslant k}} w^k |D^\alpha g(x)| < \varepsilon\right\};$$

where k is a nonnegative integer and $\varepsilon > 0$. We will find a member of 0-neighborhood base in τ_s contained in $N(V, B)$. Let k_1 be the number of multi-indices β, $\beta \leqslant \alpha$, $|\alpha| \leqslant k$, and let $c_k = (k_1!)^{k_1}$. Define the function $\chi(x)$ by

$$\chi(x) = \sup_{\substack{\phi \in B, \beta \leqslant \alpha \\ |\alpha| \leqslant k, \alpha \in R^n}} |D^{\alpha-\beta}\phi(x)|.$$

Since B is bounded in $\mathscr{K}_M$ it follows that $w^{k_2}\chi(x)$ is bounded in R^n for all positive integers k_2. Hence there exists a function ψ in $\mathscr{K}_M$ such that $\chi(x) \leqslant \psi(x)$ for all x in R^n. Let $V(k, \varepsilon, \psi) = \{f \in O_m(\mathscr{K}'_M : \mathscr{K}'_M)$: $\rho_{k,\psi}(f) < \varepsilon/c_k\}$. V is a member of 0-neighborhood base in τ, hence it is a member of 0-neighborhood base in τ_s. To complete the proof of the assertion it suffices to show that $V(k, \varepsilon, \psi)$ is contained in $N(V, B)$. Let $f \in V$ and $\phi \in B$. By the choice of ψ and Leibniz formula one has

$$\rho_k(f\phi) = \sup_{\substack{x \in R^n \\ |\alpha| \leqslant k}} w^k |D^\alpha(f\phi)(x)| \leqslant c_k \sup_{\substack{x \in R^n \\ \beta \leqslant \alpha, |\alpha| \leqslant k}} w^k |(D^\beta f)(x)\psi(x)|;$$

$$\leqslant c_k\, \rho_{k,\psi}(f) < \varepsilon,$$

i.e. $f \in N(V, B)$.

The fact that τ_b is stronger than τ_s is obvious since singleton sets in $\mathscr{K}_M$ are bounded. This completes the proof of the theorem.

Added in proof. The topologies τ_b' and τ_p of $O_c'(\mathscr{K}'_M : \mathscr{K}'_M)$ are equal; the proof will appear somewhere else.

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حول فضاءات مؤثرات الإلتفاف والضوارب في $\mathscr{K}'_M$

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خلاصة

ليكن $\mathscr{K}'_M$ الفضاء المكون من توزيعات شوارتز التي تنمو بحد أقصى مساوٍ للدالة $e^{M(kx)}$ ، حيث k عدد صحيح و M دالة دليلية . في هذا البحث ندرس عدة توپولوجيات على الفضاء $O'_c(\mathscr{K}'_M:\mathscr{K}'_M)$ المكون من فضاء ضوارب الإلتفاف على الفضاء $\mathscr{K}'_M$ ، وفضائه الثنوى $O_c(\mathscr{K}'_M:\mathscr{K}'_M)$. كذلك ندرس الفضاء $O_m(\mathscr{K}'_M:\mathscr{K}'_M)$ المكون من ضوارب الفضاء $\mathscr{K}'_M$ ونزوده بعدة توپولوجيات ، ثم نبرهن أن جميع هذه التوپولوجيات متساوية .