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An existence theorem for the Dirichlet problem for quasilinear elliptic equations in domains with corners

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ABSTRACT

The first boundary value problem for quasilinear elliptic equations in plane domains with corners is studied. By Schauder's fixed point theorem and a method introduced by Nirenberg (1953), an existence theorem for this problem is given.

1. INTRODUCTION

We continue here our investigations of boundary value problems for elliptic equations in domains with corners. We give an existence theorem for the Dirichlet problem involving a quasilinear second order elliptic equation in such domains. The method used was introduced by Nirenberg (1953) and is based on Schauder's fixed point theorem for completely continuous transformations in Banach spaces (Schauder 1930). In Section 2 we state the existence theorem. Section 3 concerns some earlier results about solutions of linear equations. To estimate the first derivatives of solutions, we use a theorem due to Radò (1926) concerning saddle functions. Section 4 contains a lemma related to the application of Radò's theorem. The proof of the existence theorem is given in Section 5. In that section we also state a theorem concerning the smoothness of the solutions.

2. EXISTENCE THEOREM

We first describe the domain $G \subset R^2$ in which we shall prove the existence theorem in Section 5. We assume that G is strictly convex and its boundary Γ consists of a finite number of curves Γ_i, $i = 1, \ldots, k$, each of which is an interior part of a curve, having strictly positive curvature. Moreover, in a neighborhood of every point of Γ_i, the boundary is given by $x = x(s)$ and $y = y(s)$, where s is the arc length measured from one of the end points of Γ_i, and $x(s)$ and $y(s)$ are $C_{2+\alpha}$-functions, $0 < \alpha < 1$. In G we consider the Dirichlet problem

$$A(x, y, z, p, q)z_{xx} + B(x, y, z, p, q)z_{xy} + C(x, y, z, p, q)z_{yy} = 0 \qquad (2.1)$$

$$z|_{\Gamma} = \phi \qquad (2.2)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. We make the following assumptions.

(i) A, B and C are defined for $(x, y) \in G$ and all z, p and q. For every positive number K and for $(x, y) \in G$ and $|z|, |p|, |q| \leqslant K$ the coefficients A, B and C satisfy the following conditions (a) and (b).

(a) A Hölder condition in x, y, z, p and q with coefficients $H(K)$ and exponent $\beta(K)$ depending only on K.

(b) The inequality

$$m(K)(\xi^2 + \eta^2) \leqslant A\xi^2 + B\xi\eta + C\eta^2 \leqslant M(K)(\xi^2 + \eta^2) \tag{2.3}$$

for all real ξ and η, where $m(K)$ and $M(K)$ depend only on K.

(ii) ϕ is continuous on Γ and belongs to $C_{2+\alpha}$ on Γ_i, $i = 1, \ldots, k$.

Under these assumptions we prove the following:

Existence theorem. There exists a solution $z \in C_{2+\sigma}(G) \cap C_{1+\delta}(\bar{G})$ of (2.1)–(2.2), where $0 < \delta < 1$ and $0 < \sigma < \alpha$. Moreover, any solution $z \in C_1(\bar{G})$ satisfies the inequality $\|z\|_{1+\delta} \leqslant \bar{H}$.

This theorem will be proved in Section 5, the idea of the proof being as follows: Choosing a function z from some class in a Banach space (to be specified later), we consider the linear equation

$$A(x, y, z, p, q)u_{xx} + B(\cdots)u_{xy} + C(\cdots)u_{yy} = 0, \tag{2.4}$$

and the boundary condition

$$u|_\Gamma = \phi. \tag{2.5}$$

This defines a mapping $z \to u$ of that class into that Banach space. Thus the problem of determining a solution of (2.1)–(2.2) is now reduced to the problem of finding a fixed point of that mapping.

3. PROPERTIES OF SOLUTIONS OF EQUATIONS THAT ARE LINEAR

In this section we state some earlier results concerning solutions of the Dirichlet problem for *linear* elliptic equations. These results will be used in the proof of the main theorem of this paper. In a bounded domain $\Omega \subset R^2$ with boundary Γ we consider the Dirichlet problem

$$a_{ij}(x)u_{ij} + a_i(x)u_i + a(x)u = f(x) \tag{3.1}$$

$$u = \phi \quad \text{on } \Gamma. \tag{3.2}$$

Here $x = (x_1, x_2)$, $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i\, \partial x_j$ and we use the summation convention. We make the following assumption:

(A1) The coefficients of (3.1) belong to $C_{s+\alpha}(\bar{\Omega})$, where $s \geqslant 0$ is an integer, $0 < \alpha < 1$, and $\bar{\Omega}$ is the closure of Ω; *cf.* Agmon *et al.* (1959) for the definition of $C_{s+\alpha}$.

A known result is as follows (Agmon *et al.* (1959): Let u be a solution of (3.1), (3.2) and let (A1) be satisfied. Furthermore, assume that Γ can be represented parametrically by $C_{s+2+\alpha}$-functions and $\phi \in C_{s+2+\alpha}(\Gamma)$. Then $u \in C_{s+2+\alpha}(\bar{\Omega})$.

If Γ has corners, this result may not hold. We now state another assumption.

(A2) Γ consists of a finite number of curves Γ_i, $i=1,\ldots,k$, of class $C_{s+2+\alpha}$. Γ_i intersects Γ_{i+1} at O_i, making an interior angle γ_i, $i=1,\ldots,k$; here $\Gamma_{k+1}=\Gamma_i$. Assume also that ϕ is continuous on Γ and of class $C_{s+2+\alpha}(\Gamma_i)$.

From Agmon *et al.* (1959), it follows, under assumptions (A1) and (A2), that the solution u of (3.1), (3.2) belongs to $C_{s+2+\alpha}(\Omega_1)\cap C_0(\bar{\Omega})$, where Ω_1 is any compact subdomain of $\bar{\Omega}$ with positive distances from the corners. For the smoothness of the solutions near the corners we state the following theorems:

Theorem 1. Under assumptions (A1) and (A2) any solution of (3.1), (3.2) belongs to $C_v(\bar{\Omega}_i)$, where Ω_i is a small neighborhood of the corner point O_i, and $v=\min(s+2+\alpha, (\pi/\omega_i)-\varepsilon)$ with arbitrarily small $\varepsilon>0$, and

$$\omega_i=\arctan\{[a_{11}(O_i)a_{22}(O_i)-a_{12}^2(O_i)]^{1/2}/[a_{22}(O_i)\cot\gamma_i-a_{12}(O_i)]\}, \quad (3.3)$$

$i=1,2,\ldots,k$. *Cf.* (3.1).

Note that $\gamma_i=\omega_i$ if the leading part of (3.1) at O_i is the Laplacian. This result was proved in Azzam (1980a) for $\omega<\pi$ and domains in R^n with $n\geqslant 2$ (see also Azzam (1980b, c, 1983)). Moreover, it was also proved in Azzam (1980a, b, c) that there exist positive numbers τ and τ_0, both less than 1, such that if $\omega<\pi$, then for any second derivative u_{ij}, $i,j=1,2$, of a solution of (3.1), (3.2) we have $r^\tau u_{ij}\in C_{\tau_0}(\bar{\Omega})$, where $r(x)$ is the distance of x from a closest corner point on Γ. Note that the result of Azzam (1980a) implies that the first partial derivatives of u satisfy a Hölder condition with exponent $\mu=v-1$, $0<\mu<1$ and that

$$\|u\|_{1+\mu}\leqslant M'. \quad (3.4)$$

However, this bound M' is not convenient for proving the existence theorem, since M' depends, among other numbers, on the Hölder continuity properties of the coefficients of (3.1). In contrast to this situation, the following theorem gives an estimation of the form $\|u\|_{1+\delta}\leqslant M$ with the constant M being suitable in proving the existence theorem.

Theorem 2. Let $\Omega\subset R^2$ be a finite domain whose boundary consists of a finitely many C_2-curves intersecting at the points $O_1,\ldots,O_k$ and making angles less than π at these points. Let u be a solution of the Dirichlet problem

$$au_{xx}+bu_{xy}+cu_{yy}+d=0, \quad \text{in } \Omega \qquad u=\phi \quad \text{on } \partial\Omega.$$

Suppose that $u\in C_2(\Omega_1)\cap C_1(\bar{\Omega})$, where Ω_1 is any compact subdomain of $\bar{\Omega}$ with positive distance from the corner points. Furthermore, assume the following:

(i) $|a(x\ y)|, |b(x,y)|, |c(x,y)|, |d(x,y)|\leqslant K$ in $\bar{\Omega}$.

(ii) $|u_x(x,y)|, |u_y(x,y)|\leqslant K_1$ in $\bar{\Omega}$.

(iii) For every real ξ and η and $(x,y)\in\Omega$ we have

$$a(x,y)\xi^2+b(x,y)\xi\eta+c(x,y)\eta^2\geqslant m(\xi^2+\eta^2),$$

where $m>0$ is a constant.

(iv) $\phi\in C_2(\partial\Omega/\bigcup_i\{O_i\})\cap C_0(\partial\Omega)$.

Then the first derivatives of u satisfy in $\bar{\Omega}$ a Hölder condition with exponent δ and coefficient depending only on K, K_1, $\|\phi\|_2$, m and the domain Ω.

The proof is extremely technical and goes along the same lines as in Nirenberg (1953), where the smooth boundary case was considered. We omit the proof.

4. A LEMMA ON SADDLE FUNCTIONS

To estimate the first derivatives of the solution of the Dirichlet problem for the equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = 0, \tag{4.1}$$

we apply a theorem on saddle functions due to Radò (1926). For this purpose we first give the following two definitions:

Definition 1. A surface $z = z(x, y)$ with non-positive Gauss curvature is called a *saddle surface*.

Definition 2. Let $z(x, y)$ be a continuous function defined on the boundary of a domain G. The curve $\mathscr{L}$ in R^3 defined by $z(x, y)$ is said to satisfy the *triangle condition* with a constant Δ, if for any plane $z = \xi x + \eta y + \zeta$, which passes through three different points on $\mathscr{L}$, the following inequality holds:

$$\sqrt{\xi^2 + \eta^2} \leqslant \Delta. \tag{4.2}$$

For saddle functions, Radò (1926) obtained a theorem which states that if G is strictly convex and if $z(x, y)$ is a continuous saddle function in G and its boundary values define a curve $\mathscr{L} \subset R^3$ which satisfies the triangle condition with a constant Δ, then z satisfies in $\bar{G}$ a Lipschitz condition and the first derivatives z_x and z_y, which exist almost everywhere in G, are bounded in absolute value by Δ.

Radò's theorem can be applied to our present setting where we have corners. Indeed, the following lemma holds:

Lemma 1. Consider the domain G given in Section 2. Let $z(x, y)$ be continuous on Γ and of class C_2 on Γ_i, $i = 1, \ldots, k$. Then the space curve $\mathscr{L}$ defined by $z(x, y)$ satisfies the triangle condition with a constant $\Delta \leqslant K_0(\beta)$, where $\beta = \|z\|_2^{\Gamma}$.

Proof. Since the curvature of Γ_i, $i = 1, \ldots, k$, is positive, the determinant

$$\begin{vmatrix} x'(s) & y'(s) \\ x''(s) & y''(s) \end{vmatrix}$$

has a positive lower bound, say, 2κ. Let P_1 and P_2 be two points on the same curve Γ_i, and let the corresponding values of the arc length, measured from an endpoint of the curve, be s_1 and s_2, respectively, where $0 \leqslant s_1 < s_2$. From the definition of κ it follows that there exists $\rho_0 > 0$ such that for $s_2 - s_1 < \rho_0$ we have

$$\begin{vmatrix} x'(s_1) & y'(s_1) \\ x''(s_2) & y''(s_2) \end{vmatrix} > \kappa. \tag{4.3}$$

To prove the lemma, we must show that, for any plane $z = \xi x + \eta y + \zeta$ which intersects $\mathscr{L}$ in three different points P_1, P_2 and P_3, the inequality (4.2) holds with $\Delta \leqslant K_0(\beta)$. We have to consider the three points in various positions relative to each other and to the corners, and obtain (4.2) in each case. There are thirteen different cases. We consider here only three of these cases, which we denote by (C1), (C2i) and (C2ii), since each of the other cases can be treated similarly to either (C2i) or (C2ii).

(C1) P_1, P_2 and P_3 lie on the same curve Γ_i. Let s_1, s_2 and s_3 be the values of the arc length measured from an end point of the curve. Let $0 \leqslant s_1 < s_2 < s_3$ and $s_3 - s_1 \leqslant \rho_0$. We shall now determine a bound for

$$\xi = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \tag{4.4}$$

where, for simplicity, we have put $x_i = x(s_i)$, $y_i = y(s_i)$ and $z_i = z(s_i)$. Using the mean value theorem, we get

$$\xi = \begin{vmatrix} y'(s_{12}) & z'(s_{12}) \\ y''(s_{13}) & z''(s_{13}) \end{vmatrix} \Bigg/ \begin{vmatrix} x'(s_{12}^0) & y'(s_{12}^0) \\ x''(s_{13}^0) & y''(s_{13}^0) \end{vmatrix}, \tag{4.5}$$

where $s_1 < s_{12} < s_2$ and $s_1 < s_{13} < s_3$. From (4.3) we obtain $|\xi| \leqslant K_0\beta$. η may be estimated in a similar fashion.

(C2) P_1 and P_2 are on the same curve while P_3 is on a neighboring curve. We take the common corner point as an origin. Let s_1, s_2 and s_3 be the values of the arc length measured from this origin, and let $0 \leqslant s_1 \leqslant s_2$.

(i) $s_2 - s_1 > \rho_0$ and $s_3 > \rho_0$. In this case the denominator in (4.5) has a positive lower bound. Indeed, if we had

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = 0,$$

then it would follow that

$$\frac{y_3 - y_1}{x_3 - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which means that the two lines P_1P_3 and P_1P_2 would be parallel, which is false. Hence, also in this case we have $|\xi| \leqslant K_0(\beta)$.

(ii) $s_2 < \rho_0$ and $s_3 < \rho_0$. In this case we show that $\xi^2 + \eta^2 \leqslant K_0^2(\beta)$. Suppose not. Then there exist (a) and (b):

(a) Sequences of points s_1^n, s_2^n and s_3^n, with $s_1^n < s_2^n < \rho_0$ and $s_3^n < \rho_0$.
(b) A sequence $z_n(s)$ with $|z_n(s)| + |z_n'(s)| + |z_n''(s)| \leqslant \beta$ and

$$\sqrt{\xi_n^2 + \eta_n^2} \to \infty. \tag{4.6}$$

From (b) it follows that there exists a subsequence $z_m(s)$ of $z_n(s)$ such that $z_m(s) \to z(s)$ uniformly as well as $z_m'(s) \to z'(s)$ uniformly and $|z(s)| + |z'(s)| \leqslant \beta$. At the same time, in s_1^n, s_2^n and s_3^n we can find convergent subsequences s_1^p, s_2^q and s_3^r. Let s_1^0, s_2^0 and s_3^0 be the corresponding limits. If $s_1^0 \neq s_2^0 \neq s_3^0$, then the plane passing through the points s_1^p, s_2^q and s_3^r tends to a plane passing through the three points s_1^0, s_2^0 and s_3^0. If $s_1^0 = s_2^0 \neq s_3^0$, then the plane tends to a plane passing through s_3^0 and touching

$\mathscr{L}$ in $s_1 = s_2$. In both cases the limiting planes are not perpendicular to the (x, y)-plane. This contradicts (4.6). Finally, if $s_1^0 = s_2^0 = s_3^0$, then the limit is the corner point itself. This case may be treated as case (C1), and the proof of the lemma is complete.

5. PROOF OF THE EXISTENCE THEOREM

As we had mentioned before, we use the Schauder fixed point theorem in our proof. We first prove the following:

Lemma 2. Let u be a solution of the linear elliptic equation

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = 0 \tag{5.1}$$

in G and assume that $u \in C_1(\bar{G})$ and $u|_\Gamma = \phi$, where $\phi \in C_2(\Gamma_i)$, $i = 1, \ldots, k$. Then

$$\|u\|_1 \leqslant K(\|\phi\|_2),$$

where K depends also on G.

Proof. Since (5.1) is elliptic and contains only second derivatives of u, we can apply the maximum principle to get

$$\|u\|_0 \leqslant \|\phi\|_0. \tag{5.2}$$

Furthermore, it follows from the ellipticity of (5.1) that the surface in three-dimensional space represented by $u = u(x, y)$ is a saddle surface. Indeed, multiplying (5.1) by u_{xx}, we see that

$$au_{xx}^2 + bu_{xx}u_{xy} + cu_{xy}^2 = c(u_{xy}^2 - u_{xx}u_{yy}). \tag{5.3}$$

Since the quadratic on the left side has the same sign as c, we have $u_{xx}u_{yy} - u_{xy}^2 > 0$ and the Gauss curvature of the surface is non-positive. Applying the result of Lemma 1, we conclude that in $\bar{G}$

$$\left|\frac{\partial u}{\partial x}\right| \leqslant K_0(\|\phi\|_2), \quad \left|\frac{\partial u}{\partial y}\right| \leqslant K_0(\|\phi\|_2), \tag{5.4}$$

where K_0 depends also on G. Now (5.2) and (5.4) yield

$$\|u\|_1 \leqslant K(\|\phi\|_2).$$

This proves Lemma 2.

We now consider the linear equation

$$A(x, y, z, p, q)u_{xx} + B(x, y, z, p, q)u_{xy} + C(x, y, z, p, q)u_{yy} = 0 \tag{5.5}$$

and denote by M and m the numbers $M(K)$ and $m(K)$ introduced in condition (i) of the existence theorem. Here $K = K(\|\phi\|_2)$ is obtained from Lemma 2.

Lemma 3. Let $z \in C_1(\bar{G})$, $\|z\|_1 \leqslant K$, and let $u \in C_1(\bar{G})$ be the solution of (5.5) which on Γ coincides with the given function ϕ. Then $\|u\|_1 \leqslant K$ and there exist constants $\bar{K}$ and $\delta < 1/2$ which depend only on K, m, M, $\|\phi\|_2$ and the domain G, such that

$$\|u\|_{1+\delta} \leqslant \bar{K}. \tag{5.6}$$

Proof. Substituting the function $z \in C_1$, $\|z\|_1 \leqslant K$ in the coefficients of (5.5), we get an elliptic equation. Using Lemma 2, we see that $\|u\|_1 \leqslant K$. We now apply Theorem 2 to conclude that the first derivatives of u satisfy in $\bar{G}$ a Hölder condition with coefficients M_0 and exponent $\delta < 1/2$ which depend only on M, m, K, $\|\phi\|_2$ and the domain G. Together with $\|u\|_1 \leqslant K$ we obtain (5.6). This proves the lemma.

Consider now the space $C_{1+\delta'}(\bar{G})$, $0 < \delta' < \delta$, where δ is the number in Lemma 3. In this space we take the set $S_{1+\delta'}$ of functions of class $C_2(G)$ which satisfy the inequalities

$$\|z\|_1 \leqslant K, \quad \|z\|_{1+\delta'} \leqslant \bar{\bar{K}} \tag{5.7}$$

where $\bar{\bar{K}} > K + M_0 l^{\delta-\delta'}$ and l is the diameter of G. By H and β we denote the numbers $H(K)$ and $\beta(K)$ introduced in the existence theorem, where K is the number $K(\|\phi\|_2)$ in Lemma 2. We set

$$A(x, y, z, p, q) = a(x, y),$$

$$B(x, y, z, p, q) = b(x, y),$$

$$C(x, y, z, p, q) = c(x, y).$$

Lemma 4. There exists a unique solution u of the linear equation

$$au_{xx} + bu_{xy} + cu_{yy} = 0 \tag{5.8}$$

coinciding on Γ with the given function ϕ. Moreover, there exist constants $\bar{K}$ and $\delta < 1/2$ such that any solution $u \in C_1(\bar{G})$ of the problem satisfies $\|u\|_{1+\delta} \leqslant \bar{K}$.

Proof. Since the functions p and q satisfy a Hölder condition with exponent δ' and coefficient $\bar{\bar{K}}$, and the functions A, B and C satisfy a Hölder condition with exponent β and coefficient H, the functions a, b and c satisfy in $\bar{G}$ a Hölder condition with exponent $\beta\delta'$ and coefficient depending on K, $\bar{\bar{K}}$ and H. We note that from the condition (b) of the existence theorem and from $\|z\|_1 \leqslant K$ it follows that

$$m(\xi^2 + \eta^2) \leqslant a(x, y)\xi^2 + b(x, y)\xi\eta + c(x, y)\eta^2 \leqslant M(\xi^2 + \eta^2)$$

for real ξ and η, where $M = M(K)$ and $m = m(K)$ with $K = K(\|\phi\|_2)$. We can now apply the existence theorem in Oleĭnik (1949) to conclude that there exists a unique solution of the problem which is continuous on $\bar{G}$ and of class $C_{2+\sigma}(G_1)$, where $\sigma = \min(\beta\delta', \alpha)$ and G_1 is any compact sub-domain of $\bar{G}$ with positive distance from the corner points. From Theorem 1 it follows that $u \in C_\nu(\bar{G})$, $1 < \nu < 2$, and from Theorem 2 we see that $u \in C_{1+\delta}(G)$ and there exists a number $\bar{K}$, depending only on M, m, K, $\|\phi\|_2$ and the domain G, such that

$$\|u\|_{1+\delta} \leqslant \bar{K}.$$

This proves the lemma.

Proof of the existence theorem. Let $S_{1+\delta'}$ be the class defined above. Consider the map $u = u(z)$ defined on $S_{1+\delta'}$ where u is the solution of (5.5), with $z \in S_{1+\delta'}$ substituted in the coefficients. Using Lemmas 3 and 4 we conclude that $u(z)$ maps $S_{1+\delta'}$ into itself. Moreover, this mapping is completely continuous. Indeed, the set $\{z\} \subset S_{1+\delta'}$ is mapped onto $\{u\} \subset C_{1+\delta}$ with $\|u\|_{1+\delta} \leqslant \bar{K}$, and it is known that any ball $\|u\|_{\mu_1} \leqslant \bar{K}$ is in C_{μ_2} if $\mu_1 > \mu_2$. Thus $\{u\}$ is compact in $C_{1+\delta'}$. We now prove that the mapping is continuous. For this purpose we consider the space $C^*_{2+\tau_0}(\bar{G})$ of functions from $C_1(\bar{G})$ whose second derivatives multiplied by r^τ belong to C_{τ_0}; here $0 \leqslant \tau, \tau_0 < 1$, and r is the distance from a closest corner point on Γ. In $C^*_{2+\tau_0}$ we define the norm

$$\|u\|^*_{2+\tau_0} = \|u\|_1 + \|r^\tau D^2 u\|_{\tau_0}.$$

From Theorem 1 it follows that the solution u of (5.5) belongs to this space. Consider now the set $\{z_n\} \subset S_{1+\delta'}$ and suppose that $z_n \to z \in S_{1+\delta'}$. We show that $u_n = u(z_n) \to u(z)$. Since $\|u_n\|^*_{2+\tau_0} \leqslant A_0$, this ball is compact in C^*_2. Thus from $\{u_n\}$ we can extract a subsequence $\{u_m\}$ which converges, together with its first derivatives, to a function u_0 and the corresponding derivatives, respectively. Also the sequence $\{r^\tau u''_m\}$ converges to $r^\tau u''_0 \in C_0$. Now u_m is a solution of a linear equation

$$A(x, y, z_m, p_m, q_m)(u_m)_{xx} + B(\cdots)(u_m)_{xy} + C(\cdots)(u_m)_{yy} = 0$$

where $p_m = \partial z_m/\partial x$ and $q_m = \partial z_m/\partial y$. Let G_1 be a subdomain of G which includes only points of G with distances from the corner points not less than r_0, where $r_0 < d_0$ and $2d_0$ is the smallest distance between any two corner points. We define the function

$$v_m^{xx} = \begin{cases} r_0^\tau (u_m)_{xx} & \text{if } (x, y) \in G_1 \\ r^\tau (u_m)_{xx} & \text{if } (x, y) \notin G_1 \end{cases}$$

where r is the distance to a closest corner point. The functions v_m^{xy} and v_m^{yy} are defined similarly. It is clear that these functions are continuous and that they converge to v_0^{xx}, v_0^{xy} and v_0^{yy}, respectively, where, for example

$$v_0^{xx} = \begin{cases} r_0^\tau (u_0)_{xx} & \text{if } (x, y) \in G_1 \\ r^\tau (u_0)_{xx} & \text{if } (x, y) \notin G_1. \end{cases}$$

Since

$$A(x, y, z_m, p_m, q_m) v_m^{xx} + B(\cdots) v_m^{xy} + C(\cdots) v_m^{yy} = 0,$$

and all the functions involved are continuous, we obtain in the limit

$$A(x, y, z, p, q) v_0^{xx} + B(\cdots) v_0^{xy} + C(\cdots) v_0^{yy} = 0$$

or equivalently,

$$A(x, y, z, p, q) u_{0,xx} + B(\cdots) u_{0,xy} + C(\cdots) u_{0,yy} = 0.$$

Since the solution of the linear equation is unique, we get

$$u_0 = u(z) = \lim_{m \to \infty} u(z_m).$$

Hence the mapping $u = u(z)$ is continuous.

We can now apply Schauder's fixed point theorem. Thus there exists a fixed point z_0, that is, $u(z_0) = z_0$. This is a solution of the Dirichlet problem for the quasilinear equation. As a solution of the linear equation it belongs to $C_{2+\sigma}(G) \cap C_{1+\delta}(\bar{G})$. This proves the theorem.

We conclude this section by a theorem on further smoothness properties of the solution of (2.1), (2.2).

Theorem 3. Consider the strictly convex domain $G \subset R^2$. Let the boundary of G be of class $C_{s+2+\alpha}$, except at a point O, where it has a corner of angle $\gamma > 0$; here, $s \geqslant 0$ an integer, and $0 < \alpha < 1$. Let $z(x, y)$ be a solution of (2.1)–(2.2), where

$\phi \in C_{s+2+\alpha}(\Gamma\setminus\{0\}) \cap C_0(\Gamma)$. Assume that A, B and C are defined for $(x, y) \in G$ and for all z, p and q. For every positive number K and $(x, y) \in G$ as well as $|z|, |p|, |q| \leqslant K$, the coefficients A, B and C belong to $C_{s+\alpha}(\bar{G})$ and satisfy the ellipticity condition (2.3). Then the solution z of (2.1)–(2.2) belongs to $C_{s+2+\alpha_1}(\bar{G}\setminus\{0\})$, where $\alpha_1 = \alpha$ if $s > 0$ and $\alpha_1 = \alpha\delta$ if $s = 0$ and δ is defined in Theorem 2. Let the corner point O be at the origin, and let $y = 0$, $x \geqslant 0$ be a tangent to Γ at the point O. Let

$$\omega = \arctan[(\xi_{11}\xi_{22} - \xi_{12}^2)^{1/2}/(\xi_{22}\cot\gamma - \xi_{12})],$$

where

$$\xi_{11} = A(0, 0, \phi(0, 0), \phi_x(0, 0), \phi_l(0, 0)\sin\gamma), \quad 2\xi_{12} = B(\cdots), \quad \xi_{22} = C(\cdots),$$

and ϕ_l is the derivative of ϕ in the direction $\theta = \gamma$. Then $z \in C_v(\bar{G})$, where $v = \min(s + 2 + \alpha_1, (\pi/\omega) - \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small.

The proof follows immediately from Theorem 1 concerning linear equations and from the existence theorem. Note that if Γ has more than one corner, we can investigate the smoothness of z near each corner separately, proceeding similarly as above.

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نظرية وجود لمسألة دريشلية لمعادلة إهليليجية شبه خطية في مجالات غير ملساء

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خلاصة

ندرس في هذا البحث مسألة وجود حل للمسألة الحدية الأولى لمعادلة تفاضلية جزئية شبه خطية من النوع الإهليليجي ، وذلك في مجالات غير ملساء . باستعمال بعض نظريات التحليل الرياضي أمكن إثبات وجود حل للمسألة ، كما درست بعض خصائص هذا الحل .