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Results on fixed point theorems for multivalued mappings

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ABSTRACT

Some common fixed point theorems for a family of multifunctions on a non-empty complete metric space have been proved.

1. INTRODUCTION

Much work has been done on the fixed point theorems of multifunctions and common fixed point theorems of multifunctions on metric spaces; see Fisher (1980, 1981–82, 1985), Popa (1984) and Toru (1977). The chief aim of this paper is to prove some common fixed point theorems for a family of multifunctions on a non-empty complete metric space. The method used is that used in Achari (1979) with necessary modifications as required for more general settings.

We denote by $CB(X)$ the set of all non-empty closed bounded subsets of X and by H the Hausdorff–Pompeiu metric on $CB(X)$.

Let $A, B \in CB(X)$; then we have from the well known lemma of Nadler (1969), that for each $a \in A$, there is a $b \in B$ such that

$$d(a, b) \leqslant H(A, B) \quad . \tag{1}$$

Theorem 1. Let (X, d) be a complete metric space and $F_i : X \to CB(X)$, $i = 1, 2, \ldots, m$ be mappings satisfying the condition

$$H^2(F_i x, F_j y) \leqslant \alpha \max \{d(x, y)d(x, F_i x), d(x, y)d(y, F_j y) \quad , \\ d(x, F_i x)d(y, F_j y), d(x, F_j y)d(y, F_i x)\} \tag{2}$$

for all $x, y \in X$, $i, j = 1, 2, \ldots, m$ and for some α, $0 \leqslant \alpha < 1$. Then $\{F_i\}_{i=1}^{m}$ have a common fixed point, i.e. there exists $u \in X$ such that $u \in F_i(u)$, $i = 1, 2, \ldots, m$.

Proof. If $\alpha = 0$, the proof is immediate. Let $\alpha > 0$ and $x_1 \in F_1(x_0)$. Since $\alpha > 0$, by the lemma of Nadler (1969) there exists $x_2 \in F_2 x_1$ such that

$$d(x_1, x_2) \leqslant H(F_1 x_0, F_2 x_1) \quad .$$

So

$$d^2(x_1, x_2) \leqslant H^2(F_1x_0, F_2x_1) \leqslant \alpha \max \{d(x_0, F_1x_0)d(x_0, x_1) \quad , \\ d(x_0, x_1)d(x_1, F_2x_1), d(x_0, F_1x_0)d(x_1, F_2x_1) \quad , \\ d(x_0, F_2x_1)d(x_1, F_1x_0)\} \quad ,$$

i.e.

$$d^2(x_1, x_2) \leqslant \alpha \max \{d(x_0, x_1)d(x_0, x_1), d(x_0, x_1)d(x_1, x_2)\} \quad .$$

If we take $d^2(x_0, x_1)$ as maximum, then we get

$$d^2(x_1, x_2) \leqslant \alpha d^2(x_0, x_1)$$

i.e.

$$d(x_1, x_2) \leqslant \sqrt{\alpha} d(x_0, x_1) \quad . \tag{3}$$

If we take $d(x_0, x_1)d(x_1, x_2)$ as maximum, we have

$$d^2(x_1, x_2) \leqslant \alpha d(x_0, x_1)d(x_1, x_2) \quad ,$$

i.e.,

$$d(x_1, x_2) \leqslant \alpha d(x_0, x_1) \quad . \tag{4}$$

Let $\theta = \max \{\sqrt{\alpha}, \alpha\}$; then $\theta < 1$ and (3) and (4) reduce to

$$d(x_1, x_2) \leqslant \theta d(x_0, x_1) \quad . \tag{5}$$

In general, let $x_n \in F_r(x_{n-1})$, where $n = pm + r$ with $0 < r \leqslant m$, and let

$$d(x_n, x_{n+1}) \leqslant H(F_r(x_{n-1}), F_{r+1}(x_n)) \quad .$$

We claim that $\{x_n\}$ is a Cauchy sequence. It is easy to see that

$$d(x_i, x_{i+1}) \leqslant \theta^i d(x_0, F_1x_0)$$

for $1 \leqslant i \leqslant m - 1$. Also

$$d(x_m, x_{m+1}) \leqslant H(F_m(x_{m-1}), F_{m+1}(x_m)) \leqslant \theta^m d(x_0, F_1x_0) \quad .$$

Next assume, by way of induction, that for some integer $p > m$

$$d(x_j, x_{j+1}) \leqslant \theta^j d(x_0, F_1x_0)$$

for $j = 1, 2, \ldots, p - 1$.

Let $p = qm + r$; then $x_{p+1} \in F_{r+1}(x_p)$ and

$$\begin{aligned} d(x_p, x_{p+1}) &\leqslant H(F_r(x_{p-1}), F_{r+1}(x_p)) \\ &\leqslant \theta d(x_p, x_{p-1}) \\ &\leqslant \theta^p d(x_0, F_1x_0) \end{aligned}$$

and thus we have

$$d(x_n, x_{n+1}) \leqslant \theta^n d(x_0, F_1x_0)$$

for $n = 1, 2, \ldots$. Now

$$d(x_n, x_{n+p}) \leqslant \sum_{i=n}^{n+p-1} d(x_i, x_{i-1}) \leqslant \left(\sum_{i=n}^{n+p-1} \theta^i \right) d(x_0, F_1x_0) \to 0$$

for $p \geqslant 1$. Now this shows that $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X, $x_n \to u \in X$. It can easily be shown that u is the desired common fixed point of the family $\{F_i\}_{i=1}^{m}$.

Definition. A mapping $F: (X, d) \to (CB(X), H)$ is said to be upper semi-continuous, if for each open set U in X, $\{x \in X, F(x) \subseteq U\}$ is open. Further if X is compact, then it is easily verified that $X \to CB(X)$ is upper semi-continuous iff $x_n \to x$, $y_n \in F_n(x_n)$ and $y_n \to y$ implies that $y \in F(x)$.

Theorem 2. Let X be a compact metric space and for each $\lambda \in \Lambda$, Λ being an arbitrary indexing set, let F_λ: $x \to CL(X)$ the upper semi-continuous mapping and let

$$H^2(F_\lambda x, F_\mu y) \leqslant \alpha \max \{d(x, y)d(x, F_\lambda x), d(x, y)d(y, F_\mu y) \quad , \\ d(x, F_\lambda x)d(y, F_\mu y), d(x, F_\mu y)d(y, F_\lambda x)\}$$

for all $x, y \in X$, $\lambda, \mu \in \Lambda$ and some $0 < \alpha < 1$. Then the family $\{F_\lambda\}_{\lambda \in \Lambda}$ has a simultaneous fixed point.

Proof. Let $B_\lambda = \{x \in X: x \in F_\lambda x\}$ for each $\lambda \in \Lambda$. Then $B_\lambda \neq \phi$. Since F_λ is upper semi-continuous, each B_λ is closed. Next, if $B\lambda_i$, $i = 1, 2, \ldots, m$ is a finite collection. Then by Theorem 1, $\bigcap_{i=1}^{m} B_{\lambda_i} \neq \phi$. Thus $\{B_\lambda)_{\lambda \in \Lambda}$ is a collection of non-empty closed subsets of X having the finite intersection property. Using compactness of X, $\bigcap_{\lambda \in \Lambda} B_\lambda \neq \phi$.

It is easy to see that for any $u \in \bigcap_{\lambda \in \Lambda} B_\lambda$, $u \in F_\lambda u$ for all $\lambda \in \Lambda$.

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نتائج حول مبرهنات النقطة الصامدة لتحويلات متعددة القيم

جاكاديش اتشاري
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خلاصة

لقد تم اثبات بعض مبرهنات النقطة الصامدة المعروفة لعائلة من الدوال متعددة القيم على فضاء متري تام غير خال .