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Integral formulae for holomorphic and pluriharmonic functions on circular star-shaped domains

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ABSTRACT

Let D be a bounded circular star-shaped domain with respect to $0 \in D$ in $\mathbb{C}^N$ $(N > 1)$. Let

$$A^p(D) = \left\{ f\colon f \text{ holomorphic on } D \text{ and } \|f\|_{A^p} = \left(\frac{1}{V'} \int_D |f(z)|^p \, dV_z \right)^{1/p} < \infty,\ 0 < p < \infty \right\}$$

and

$$a^p(D) = \left\{ h\colon h \text{ pluriharmonic on } D \text{ and } \|h\|_{a^p} = \left(\frac{1}{V'} \int_D |h(z)|^p \, dV_z \right)^{1/p} < \infty,\ 1 \leqslant p < \infty \right\},$$

where V' is the Euclidean volume of D and dV_z is the Euclidean volume element at $z \in D$.

Motivated by the work of Stoll (1974) we prove integral formulae for functions in A^p and a^p in case D is bounded circular star-shaped domain.

INTRODUCTION

Let D be a bounded circular star-shaped domain in $\mathbb{C}^N$ $(N > 1)$ and $0 \in D$, i.e. if $z \in D$ and $\lambda \in \mathbb{C}$ and $|\lambda| \leqslant 1$ then $\lambda z \in D$.

We denote $\mathrm{ph}(D)$ and $\mathrm{H}(D)$ the space of pluriharmonic and holomorphic functions on D respectively. For $0 < p < \infty$ we set

$$A^p = A^p(D) = \left\{ f\colon f \in \mathrm{H}(D) \text{ and } \|f\|_{A^p} = \left(\frac{1}{V'} \int_D |f(z)|^p \, dV_z \right)^{1/p} < \infty \right\}$$

or equivalently Marzuq (1984a)

$$A^p = A^p(D) = \left\{ f\colon f \in \mathrm{H}(D) \text{ and } \|f\|_{A^p} = \sup_{0<r<1} M'_p(r, f) < \infty \right\}$$

where

$$M'_p(r, f) = \left(\frac{1}{V'} \int_D |f(rz)|^p \, dV_z \right)^{1/p},$$

V' is the Euclidean volume of D and dV_z is the Euclidean volume element at $z \in D$. Also, for $p \geqslant 1$, set

$$a^p = a^p(D) = \left\{ h\colon h \in \mathrm{ph} \text{ and } \|h\|_{a^p} = \left(\frac{1}{V'} \int_D |f(z)|^p \, dV_z \right)^{1/p} < \infty \right\}$$

or equivalently we can show that as in Marzuq (1984a, Theorem 1, p. 208)

$$a^p = a^p(D) = \left\{ h\colon h \in \mathrm{ph} \text{ and } \|h\|_{a^p} = \sup_{0<r<1} M_p'(r, h) < \infty \right\}.$$

In case D has a Bergman–Silov boundary, the space ph^p $(1 \leqslant p < \infty)$ is defined on D by

$$\mathrm{ph}^p = \mathrm{ph}^p(D) = \left\{ h\colon h \in \mathrm{ph}(D) \text{ and } \|h\|_{\mathrm{ph}^p} = \sup_{0<r<1} M_p(r, f) < \infty \right\}$$

where

$$M_p(r, f) = \left(\frac{1}{V} \int_b |h(rt)|^p \, ds_t \right)^{1/p}.$$

V is the Euclidean volume of b and ds_t is the Euclidean volume element at $t \in b$. When $D = \Delta = \{z \in \mathbb{C}\colon |z| < 1\}$ the space $\mathrm{ph}^p(\Delta)$ will be denoted by h^p.

On a bounded circular star-shaped domain there exists a complete orthonormal system of complex homogeneous polynomials $\{\psi_{kv}\}$, $v = 1, 2, \ldots, m_N = \binom{N+k-1}{k}$, $k = 0, 1, 2, \ldots$, (Hua 1963, p. 79), normalized on D.

Let D be bounded circular star-shaped with respect to $0 \in D$, the Bergman kernel function is defined by

$$K(z, \bar{\xi}) = \sum_{k,v} \psi_{kv}(z) \overline{\psi_{kv}(\xi)}.$$

Here

$$\sum_{k,v} = \sum_{k=0}^{\infty} \sum_{v=1}^{m_k}.$$

In this paper we prove integral formulae extending some results of Stoll (1974) to A^p and a^p spaces.

PRELIMINARY RESULTS

The following lemmas are needed in the proof of theorems.

Lemma 1 (Marzuq 1984b). Let D be a bounded circular domain. Then any holomorphic function on D has a Fourier series expansion

$$f(z) = \sum c_{kv}(f) \psi_{kv}(z), \qquad c_{kv}(f) = \lim_{r \to 1} \int_D f_r(z) \overline{\psi_{kv}(z)} \, dV_z \tag{1.1}$$

where the series converges absolutely and uniformly on compact subsets of D. Here f_r is the slice function defined by $f_r(z) = f(rz)$, $z \in \bar{D}$.

Corollary 1. For $f \in A^p$ $(1 \leqslant p < \infty)$,

$$c_{kv}(f) = \int_D f(z) \overline{\psi_{kv}(z)} \, dV_z, \tag{1.2}$$

where $c_{kv}(f)$ is given by (1.1).

The proof follows by using Hölder's inequality for $1 \leqslant p < \infty$.

Lemma 2. Let D be bounded circular star-shaped domain with respect to $0 \in D$. Then the series $\sum_{k,v} \psi_{kv}(z)\overline{\psi_{kv}(\xi)}$ converges uniformly to $K(z, \bar{\xi})$ on $G \times \bar{D}$ where G is a compact subset of D.

Proof. See Bell (1969, Proposition, p. 329).

SOME PROPERTIES OF $a^p(D)$ SPACES

Let D be a star-shaped domain. Then it is well known that D is simply connected, therefore every pluriharmonic function h on D is the real part of a holomorphic function on D (Vladimirov 1966, p. 44).

We have the following theorem:

Theorem 1. Let D be a bounded symmetric domain. Then $\mathrm{ph}^p \subset a^p$ $(p \geqslant 1)$.

Proof. Let $h \in \mathrm{ph}^p$ $(1 \leqslant p < \infty)$. Then $|h|^p$ $(p \geqslant 1)$ is plurisubharmonic on D, so by Hahn & Mitchell (1969),

$$\frac{1}{V}\int_D |h(rz)|^p \, dV_z \leqslant \frac{1}{V}\int_D \left(\int_b p(z,t)|h(rt)|^p \, ds_t\right).$$

Fubini's theorem and a corollary (Stoll 1977, p. 195) applied on the right side of the above inequality gives $h \in a^p$.

Next we generalize the theorem of Riesz (see Duren 1970, p. 54), to a^p spaces. Stoll (1974) extended this theorem to ph^p spaces. In the next theorem and in the section on integral formulae (following) we will assume that D is bounded circular star-shaped domain with respect to $0 \in D$. We have the following theorem:

Theorem 2. Let $u \in a^p$ $(1 \leqslant p < \infty)$. Then its pluriharmonic conjugate v is also in a^p and

$$M_p'(r, v) \leqslant C_p M_p'(r, u), \qquad 0 \leqslant r < 1.$$

Proof. Let

$$M_{1,p}^p(r, u_z) = \frac{1}{2\pi}\int_0^{2\pi} u_z(re^{i\theta})^p \, d\theta,$$

where $u_z(re^{i\theta}) = u(re^{i\theta}z)$. Integrate over D, we get

$$M_p'^p(r, u_z) = \frac{1}{V'}\int_D M_{1,p}^p(r, u_z) \, dV_z$$

by using Fubini's theorem, the circularity of D and the circular invariance of the measure dV_z. But $M_{1,p}^p(r, u_z)$ is monotone increasing in r for all $z \in D$ since $|u_z|^p$ is subharmonic for $p \geqslant 1$. Thus by the monotone convergence theorem

$$V'\|u\|_p^p = \int_D \|u_z\|_{1,p}^p \, dV_z, \qquad \text{where} \quad \|u_z\|_{1,p}^p = \lim_{r \to 1} M_{1,p}^p(r, u_z).$$

Thus, $u \in a^p$ implies $u_z \in h^p(\Delta)$ for almost all $z \in D$.

Now by the same method used in Stoll (1974) using D instead of b we get Theorem 2.

INTEGRAL FORMULAE FOR $A^p(D)$ AND $a^p(D)$ SPACES

We have the following theorem which gives a result similar to a corollary of Stoll's (1977, p. 193).

Theorem 3. Let $f \in A^p$ $(p \geqslant 1)$. Then f has the representation

$$f(z) = \int_D K(z, \bar{\xi}) f(\xi)\, dV_\xi, \qquad z \in D \tag{3.1}$$

and

$$f(z) = \int_D T(z, \bar{\xi}) f(\xi)\, dV_\xi, \qquad z \in D \tag{3.2}$$

where $T(z, \bar{\xi}) = |K(z, \bar{\xi})|^2 / K(z, z)$.

Proof. By Lemma 1, Corollary 1 and Lemma 2

$$\begin{aligned} f(z) &= \sum_{k,v}^{\infty} c_{kv}(f) \psi_{kv}(z) \\ &= \int_D f(\xi) \left(\sum_{k,v}^{\infty} \psi_{kv}(z) \overline{\psi_{kv}(\xi)} \right) dV_\xi \\ &= \int_D K(z, \bar{\xi}) f(\xi)\, dV_\xi \end{aligned}$$

which gives (3.1).

Let

$$g(\xi) = \frac{f(\xi) K(\xi, \bar{z})}{k(z, \bar{z})}.$$

Since $K(\xi, \bar{z})/K(z, z)$ is holomorphic with respect to ξ in $\bar{D}$ for fixed $z \in D$ (see Lemma 2), and g is homomorphic in ξ, then $g \in A^p$ for $p \geqslant 1$. Then (3.1) gives (3.2) when f is replaced by g.

Also we have the following theorems:

Theorem 4. Let $u \in a^p$ $(1 < p < \infty)$. Then

$$u(z) = \int_D l(z, \xi) u(\xi)\, dV_\xi,$$

where $l(z, \xi) = 2 \operatorname{Re} K(z, \bar{\xi}) - 1/V$.

Proof. Let $u \in a^p$, $1 < p < \infty$, and let v be its pluriharmonic conjugate normalized by $v(0) = 0$. By Theorem 2, $v \in a^p$ and hence $f = u + iv \in A^p$, $1 < p < \infty$. Now,

$$u(z) = \tfrac{1}{2}(f(z) + \bar{f}(z)). \tag{3.3}$$

Since $f \in A^p$, by (3.1) substituted in (3.3)

$$u(z) = \frac{1}{2}\left[\int_D K(z, \bar{\xi}) f(\xi)\, dV_\xi + \overline{\int_D K(z, \bar{\xi}) f(\xi)\, dV_\xi} \right]. \tag{3.4}$$

Add and subtract

$$\int_D \bar{f}(\xi)K(z,\bar{\xi})\,dV_\xi + \int_D f(\xi)\overline{K(z,\bar{\xi})}\,dV_\xi$$

to the right side of (3.4). It can be shown that

$$\int_D \bar{f}(\xi)K(z,\bar{\xi})\,dV_\xi = \bar{f}(0) = u(0) \tag{3.5}$$

so that

$$u(z) = 2\int_D (\operatorname{Re} K(z,\bar{\xi}))u(\xi)\,dV_\xi - u(0). \tag{3.6}$$

Setting $z=0$ in (3.1), gives

$$u(0) = \frac{1}{V}\int_D f(\xi)\,dV_\xi, \tag{3.7}$$

since $K(0,\bar{\xi}) = 1/V$. Hence (3.6) and (3.7) give Theorem 4.

Theorem 5. Let $f \in A^p$ $(1 < p < \infty)$ with $f(0)$ real. Then

$$f(z) = \int_D M(z,\bar{\xi})u(\xi)\,dV_\xi,$$

where $M(z,\bar{\xi}) = 2K(z,\bar{\xi}) - 1/V$ and $u = \operatorname{Re} f$.

Proof. By (3.3), (3.7), (3.1), and (3.5) we have

$$\begin{aligned}\int_D M(z,\bar{\xi})u(\xi)\,dV_\xi &= \int_D 2K(z,\bar{\xi})u(\xi)\,dV_\xi - \frac{1}{V}\int_D u(\xi)\,dV_\xi \\ &= \int_D 2K(z,\bar{\xi})\tfrac{1}{2}[f(\xi)+\bar{f}(\xi)]\,dV_\xi - u(0) \\ &= f(z) + \bar{f}(0) - u(0) \\ &= f(z)\end{aligned}$$

since by assumption $f(0)$ is real.

Corollary 2. If $u \in a^p$, $1 < p < \infty$, then the pluriharmonic conjugate v of u with $v(0)=0$ is given by

$$v(z) = \int_D N(z,\xi)u(\xi)\,dV_\xi,$$

where $N(z,\xi) = 2\operatorname{Im} K(z,\bar{\xi})$.

Proof. Let $u \in a^p$. Then by Theorem 2, $v \in a^p$ for $1 < p < \infty$. Hence $f = u + iv \in A^p$.

Thus by Theorem 5

$$u(z) + iv(z) = \int_D M(z, \bar{\xi}) u(\xi)\, dV_\xi$$

$$= \int_D \left(2K(z, \bar{\xi}) - \frac{1}{V}\right) u(\xi)\, dV_\xi,$$

so that

$$v(z) = \int_D 2 \operatorname{Im} K(z, \bar{\xi}) u(\xi)\, dV_\xi$$

$$= \int_D N(z, \bar{\xi}) u(\xi)\, dV_\xi.$$

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الصيغة التكاملية لدوال عقدية ولدوال متعددة التوافق في مجال ذي شكل نجمي دائري

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خلاصة

لقد حث عمل ستول (١٩٧٤) على ايجاد صيغ تكاملية للدوال في $A^p(D)$ أي فضـاء الدوال التحليلية و $a^p(D)$ أي فضاء الدوال المتعددة التوافق ، حيث D مجال ذو شكل نجمي دائري بالنسبة إلى $0 \in D$ في فضاء مركب ذي N بعدا $\mathbb{C}^N (N > 1)$.