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The radii of starlikeness and convexity of order alpha of certain Nevanlinna analytic functions

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ABSTRACT

We determine the radius of starlikeness of order alpha and the radius of convexity of order alpha of analytic functions having the form $\int_{-1}^{1} d\mu(t)/(z-t)$ and $\int_{-1}^{1} z\, d\mu(t)/(1-tz)$, where $\mu(t)$ is a probability measure on $[-1, 1]$.

Let N_1 denote the class of Nevanlinna analytic functions

$$f(z) \equiv \int_{-1}^{1} \frac{d\mu(t)}{z-t}, \quad z \notin [-1, 1], \tag{1}$$

where $\mu(t)$ is a probability measure on $[-1, 1]$. If we replace z by $1/z$ in (1), then we obtain the class N_2 of associated functions

$$\varphi(z) \equiv f\left(\frac{1}{z}\right) \equiv \int_{-1}^{1} \frac{z\, d\mu(t)}{1-tz}, \quad z \notin \mathbb{R}\backslash(-1, 1). \tag{2}$$

We have found the radii of starlikeness and convexity of the classes N_1 and N_2 (Todorov 1980; Reade & Todorov 1981). Now we solve the general problem of finding the radii of starlikeness and convexity of arbitrary order α, $0 \leqslant \alpha < 1$, for the classes N_1 and N_2.

Theorem 1. Let $r_s(\alpha)$ denote the radius of starlikeness of order α, $0 \leqslant \alpha < 1$, of the class N_2, and let $\alpha_0 = 0{\cdot}2695\ldots$ be the root of the equation

$$4x^3 - 4x + 1 = 0 \tag{3}$$

in the interval $0 < x < 1$.

Then (i) for $0 \leqslant \alpha < \alpha_0$, the radius $r_s(\alpha)$ is equal to the root in the interval $0 < r < 1$ of the equation

$$(2\alpha-1)(2\alpha+1)^3 r^6 - (48\alpha^4 - 48\alpha^3 - 80\alpha^2 + 124\alpha - 35) r^4 + (48\alpha^4 - 144\alpha^3 + 64\alpha^2 - 12\alpha + 9) r^2 + (2\alpha-1)(3-2\alpha)^3 = 0; \tag{4}$$

(ii) for $\alpha_0 \leqslant \alpha \leqslant \sqrt{2}/2$, the radius is

$$r_s(\alpha) = \sqrt{\frac{1-\alpha}{1+\alpha}}; \tag{5}$$

(iii) for $\sqrt{2}/2 \leqslant \alpha < 1$, the radius is

$$r_s(\alpha) = \frac{1-\alpha}{\alpha}, \tag{6}$$

i.e. for each function $\varphi \in N_2$, in the disc $|z| \leqslant r_s(\alpha)$ the inequality

$$\operatorname{Re} \frac{z\varphi'(z)}{\varphi(z)} \geqslant \alpha, \quad 0 \leqslant \alpha < 1, \tag{7}$$

holds, where the equality holds only for the following extremal functions:

(a) in the case $0 \leqslant \alpha < \alpha_0$, for the functions

$$\varphi_1(z) \equiv \frac{z}{1-z^2} \left\{ 1 - \frac{z \cos \theta_s(\alpha)}{r_s(\alpha)[1 + \sin \theta_s(\alpha)]} \right\}, \tag{8}$$

and

$$\varphi_2(z) \equiv \frac{z}{1-z^2} \left\{ 1 + \frac{z \cos \theta_s(\alpha)}{r_s(\alpha)[1 + \sin \theta_s(\alpha)]} \right\} \tag{9}$$

at the two *critical points*

$$z = r_s(\alpha) e^{\pm i\theta_s(\alpha)}, \tag{10}$$

and

$$z = r_s(\alpha) e^{\pm i[\pi - \theta_s(\alpha)]}, \tag{11}$$

respectively, where $0 < \theta_s(\alpha) < \pi/2$ and $x = \sin \theta_s(\alpha)$ is the root in the interval (0, 1) of the equation

$$(2\alpha-1)^2(2\alpha+1)(2\alpha-3)x^3 + 16\alpha(\alpha-1)(2\alpha-1)x^2 + (24\alpha^2 - 24\alpha + 5)x + 2(2\alpha-1) = 0; \tag{12}$$

(b) in the case $\alpha_0 \leqslant \alpha < \sqrt{2}/2$, for the function

$$\varphi_3(z) \equiv \frac{z}{1-z^2} \tag{13}$$

at the two *critical points* $z = \pm i r_s(\alpha)$;

(c) in the case $\alpha = \sqrt{2}/2$, for the function (13) at the two *critical points* $z = \pm i(\sqrt{2} - 1)$, and for the functions

$$\varphi_4(z) \equiv \frac{z}{1+z} \tag{14}$$

and

$$\varphi_5(z) \equiv \frac{z}{1-z} \tag{15}$$

at the *critical points* $z = \sqrt{2} - 1$ and $z = -(\sqrt{2} - 1)$, respectively;

(d) in the case $\sqrt{2}/2 < \alpha < 1$, for the functions (14) and (15) at the *critical points* $z = r_s(\alpha)$ and $z = -r_s(\alpha)$, respectively.

Remark. The extremal functions φ_j $(j=1,2,3,4,5)$ have the form (2) with μ concentrated in $\{\pm 1\}$, i.e. they have the form

$$\varphi_j(z) \equiv \frac{\gamma z}{1+z} + \frac{(1-\gamma)z}{1-z} \in N_2 \quad (j=1,2,3,4,5)$$

for suitable $\gamma = \gamma_j$ in $[0, 1]$.

Proof. A) In Reade & Todorov (1981) we determined for fixed $z = re^{i\theta}$, $0<r<1$, $-\pi \leqslant \theta \leqslant \pi$, the *minimum* of the functional

$$m(z) \equiv \min \operatorname{Re} \frac{z\varphi'(z)}{\varphi(z)} \tag{16}$$

over the class N_2. We found

1) if $0 \leqslant |\theta| \leqslant (\pi/2) - 2\arctan r$, then

$$m(z) = \frac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2} > \frac{1}{2}, \tag{17}$$

where the *minimum* (17) is attained only by the function (14);

2) if $(\pi/2) + 2\arctan r \leqslant |\theta| \leqslant \pi$, then

$$m(z) = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} > \frac{1}{2}, \tag{18}$$

where the *minimum* (18) is attained only by the function (15);

3) if $(\pi/2) - 2\arctan r < |\theta| < (\pi/2) + 2\arctan r$, then

$$m(z) = \frac{1 + r\cos\theta}{1 + 2r\cos\theta + r^2} + \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2} - \frac{1 + |\sin\theta|}{2|\sin\theta|}, \tag{19}$$

where the *minimum* is attained only by the functions having the form

$$\varphi(z) \equiv \frac{z}{1-z^2}\left\{1 - \frac{z\cos\theta}{r(1+|\sin\theta|)}\right\}. \tag{20}$$

B) Now we shall find the minimum value of r, $0<r<1$, for which

$$m(z) = \alpha, \quad 0 \leqslant \alpha < 1. \tag{21}$$

In cases 1) and 2) we can obtain, from (17) and (18), the *minimum* radius in the form with

$$r = r(\alpha) = \frac{1-\alpha}{\alpha}, \quad \tfrac{1}{2} < \alpha < 1, \tag{22}$$

with the extremal functions (14) and (15) at the points $z = r(\alpha)$ and $z = -r(\alpha)$, respectively.

In the case 3) we use the following method for solving this problem. From (19) and (21) we obtain

$$m(z) - \alpha = -\frac{h(\theta,\rho)}{2|\sin\theta|[(1-\rho)^2 + 4\rho\sin^2\theta]} = 0, \tag{23}$$

where

$$h(\theta, \rho) \equiv [(\beta+1)|\sin\theta| + 1]\rho^2 - 2\cos 2\theta[(\beta-1)|\sin\theta| + 1]\rho + 1 - (3-\beta)|\sin\theta| = 0, \quad (24)$$

where

$$\beta = 2\alpha, \quad \rho = r^2. \quad (25)$$

It is clear from (24) that it is sufficient to consider $0 < \rho < 1$, $(\pi/2) - 2\arctan\sqrt{\rho} < \theta < (\pi/2) + 2\arctan\sqrt{\rho}$, only. Then the *minimum* of the function $\rho = \rho(\theta)$ can be obtained from (24) via a differentiation with respect to θ, keeping in mind that $\rho'(\theta) = 0$. The condition $\partial h/\partial\theta = 0$ yields the two equations

$$\cos\theta = 0 \quad (26)$$

and

$$(\beta+1)\rho^2 + 2[6(\beta-1)\sin^2\theta + 4\sin\theta - (\beta-1)]\rho - (3-\beta) = 0. \quad (27)$$

Now the elimination of θ from (24) and (26) and from (24) and (27) yields the value

$$\rho = \frac{2-\beta}{2+\beta}, \quad 0 \leqslant \beta < 2, \quad (28)$$

and the equation

$$k(\beta, \rho) \equiv (\beta-1)(\beta+1)^3\rho^3 - (3\beta^4 - 6\beta^3 - 20\beta^2 + 62\beta - 35)\rho^2 + (3\beta^4 - 18\beta^3 + 16\beta^2 - 6\beta + 9)\rho + (\beta-1)(3-\beta)^3 = 0, \quad (29)$$

for $0 \leqslant \beta < 2$, $\beta \neq 1$, respectively. If we eliminate ρ from (24) and (27), we obtain, with $x = \sin\theta$,

$$l(\beta, x) \equiv (\beta-1)^2(\beta+1)(\beta-3)x^3 + 4\beta(\beta-1)(\beta-2)x^2 + (6\beta^2 - 12\beta + 5)x + 2(\beta-1) = 0, \quad (30)$$

for $0 \leqslant \beta < 2$, $\beta \neq 1$.

Now by the transformation

$$y = \frac{x}{1-x} \quad (31)$$

we map the interval (0, 1) onto the interval $(0, +\infty)$, and from (30) and (31) we obtain the equation

$$(1+y)^3 l\left(\beta, \frac{y}{1+y}\right) \equiv \beta(\beta^3 - 4\beta + 2)y^3 + 2(2\beta^3 - 5\beta + 2)y^2 + (6\beta^2 - 6\beta - 1)y + 2(\beta-1) = 0, \quad (32)$$

for $0 \leqslant \beta < 2$, $\beta \neq 1$. Let $\beta_0 = 0{\cdot}5390\ldots$ and $\beta_1 = 1{\cdot}675\ldots$ be the roots in the intervals (0, 1) and (1, 2), respectively, of the equation

$$x^3 - 4x + 2 = 0. \quad (33)$$

Then by investigating the signs of the coefficients of Equation (32) we conclude that for $\beta_0 \leqslant \beta < 1$ and for $\beta_1 \leqslant \beta < 2$, Equation (32) has no positive roots, i.e. for these β Equation (30) has no roots in the interval (0, 1); for $0 \leqslant \beta < \beta_0$ and for $1 < \beta < \beta_1$, Equation (32) has only one positive root, i.e. for these β Equation (30) has only one root in the interval (0, 1).

Similarly, by the transformation (31), with $x = \rho$ in (29), we obtain the equation

$$(1+y)^3 k\left(\beta, \frac{y}{1+y}\right) \equiv 16(1-\beta)y^3 - 4(14\beta^2 - 22\beta + 7)y^2$$
$$+ 4(3\beta^3 - 23\beta^2 + 39\beta - 18)y + (\beta - 1)(3-\beta)^3 = 0, \quad (34)$$

for $0 \leqslant \beta < 2$, $\beta \neq 1$. Again by investigating the signs of the coefficients of (34) we conclude that: for $0 \leqslant \beta < 2$, $\beta \neq 1$, Equation (34) has only one positive root, i.e. Equation (29) has only one root in the interval (0, 1).

C) Now we shall compare the root in the interval (0, 1) of Equation (29) and the values (28) and (22), keeping in mind (25). The *minimum* of these three magnitudes will be the radius of starlikeness $r_s(\beta/2)$ of the class N_2. If we use (28) in (29), we obtain

$$(2+\beta)^3 k\left(\beta, \frac{2-\beta}{2+\beta}\right) \equiv 8(4-\beta)(\beta^3 - 4\beta + 2)^2, \quad 0 \leqslant \beta < 2,\ \beta \neq 1. \quad (35)$$

Thus, from (35) and by the direct comparison of (28) and (22), keeping in mind (25), we obtain the assertions (i), (ii) and (iii) in Theorem 1. Now the assertions (a), (b), (c) and (d) for the extremal functions and the *critical points* in Theorem 1 follow from steps A) and B).

This completes the proof of Theorem 1.

If we replace z by $1/z$ in Theorem 1, we get the following result for the class N_1.

Theorem 2. Let $R_s(\alpha)$ denote the radius of starlikeness of order $-\alpha$, $0 \leqslant \alpha < 1$, of the class N_1, and let $\alpha_0 = 0{\cdot}2695\ldots$ be the root of Equation (3) in the interval (0, 1).

Then (i) for $0 \leqslant \alpha < \alpha_0$, the radius $R_s(\alpha)$ is equal to the root in the interval $(1, +\infty)$ of the equation

$$(2\alpha - 1)(3 - 2\alpha)^3 R^6 + (48\alpha^4 - 144\alpha^3 + 64\alpha^2 - 12\alpha + 9)R^4$$
$$- (48\alpha^4 - 48\alpha^3 - 80\alpha^2 + 124\alpha - 35)R^2 + (2\alpha - 1)(2\alpha + 1)^3 = 0; \quad (36)$$

(ii) for $\alpha_0 \leqslant \alpha \leqslant \sqrt{2}/2$, the radius is

$$R_s(\alpha) = \sqrt{\frac{1+\alpha}{1-\alpha}}; \quad (37)$$

(iii) for $\sqrt{2}/2 \leqslant \alpha < 1$, the radius is

$$R_s(\alpha) = \frac{\alpha}{1-\alpha}, \quad (38)$$

i.e. for each function $f \in N_1$, in the disc $|z| \geqslant R_s(\alpha)$ the inequality

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \leqslant -\alpha, \quad 0 \leqslant \alpha < 1, \quad (39)$$

holds, where the equality holds only for the following extremal functions:

(a) in the case $0 \leqslant \alpha < \alpha_0$, for the functions

$$f_1(z) \equiv \frac{1}{z^2 - 1}\left\{z - \frac{R_s(\alpha)\cos\theta_s(\alpha)}{1 + \sin\theta_s(\alpha)}\right\}, \quad (40)$$

and

$$f_2(z) \equiv \frac{1}{z^2-1}\left\{z + \frac{R_s(\alpha)\cos\theta_s(\alpha)}{1+\sin\theta_s(\alpha)}\right\} \tag{41}$$

at the two *critical points*

$$z = R_s(\alpha)e^{\pm i\theta_s(\alpha)} \tag{42}$$

and

$$z = R_s(\alpha)e^{\pm i[\pi-\theta_s(\alpha)]}, \tag{43}$$

respectively, where $0 < \theta_s(\alpha) < \pi/2$ and $x = \sin\theta_s(\alpha)$ is the root of Equation (12) in the interval (0, 1);

(b) in the case $\alpha_0 \leqslant \alpha < \sqrt{2}/2$, for the function

$$f_3(z) \equiv \frac{z}{z^2-1} \tag{44}$$

at the two *critical points* $z = \pm iR_s(\alpha)$;

(c) in the case $\alpha = \sqrt{2}/2$, for the function (44) at the two *critical points* $z = \pm i(\sqrt{2}+1)$, and for the functions

$$f_4(z) \equiv \frac{1}{z+1} \tag{45}$$

and

$$f_5(z) \equiv \frac{1}{z-1} \tag{46}$$

at the *critical points* $z = \sqrt{2}+1$ and $z = -(\sqrt{2}+1)$, respectively;

(d) in the case $\sqrt{2}/2 < \alpha < 1$, for the functions (45) and (46) at the *critical points* $z = R_s(\alpha)$ and $z = -R_s(\alpha)$, respectively.

Remark. The extremal functions f_j $(j = 1, 2, 3, 4, 5)$ have the form (1) with μ concentrated in $\{\pm 1\}$, i.e. they have the form

$$f_j(z) \equiv \frac{\gamma}{z+1} + \frac{1-\gamma}{z-1} \in N_1 \quad (j = 1, 2, 3, 4, 5)$$

for suitable $\gamma = \gamma_j$ in $[0, 1]$.

In particular, for $\alpha = 0$ and $\alpha = 1/2$, Theorems 1 and 2 yield the corresponding results in Todorov (1980) and Reade & Todorov (1981).

Further, to obtain the radius of convexity of order alpha of the class N_2, we use the results in Robertson (1936), Reade & Todorov (1981) and Todorov (1983a).

Theorem 3. The radius of convexity $r_c(\alpha)$ of order α, $0 \leqslant \alpha < 1$, of the class N_2 is

$$r_c(\alpha) \equiv \sqrt{\frac{\sqrt{\alpha^2+8}-\alpha-2}{\sqrt{\alpha^2+8}-\alpha+2}}, \quad 0 \leqslant \alpha < \frac{1}{\sqrt{3}}, \tag{47}$$

$$r_c\left(\frac{1}{\sqrt{3}}\right) \equiv 2-\sqrt{3}, \quad \alpha = \frac{1}{\sqrt{3}}, \tag{48}$$

$$r_c(\alpha) \equiv \frac{1-\alpha}{1+\alpha}, \quad \frac{1}{\sqrt{3}} < \alpha < 1, \tag{49}$$

i.e. for each function $\varphi \in N_2$, in the disc $|z| \leqslant r_c(\alpha)$ the inequality

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) \geqslant \alpha, \quad 0 \leqslant \alpha < 1, \tag{50}$$

holds, where the equality holds only for the following extremal functions:

1) in the case $0 \leqslant \alpha < 1/\sqrt{3}$, for the function (13) at the *critical points* $z = \pm i r_c(\alpha)$;

2) in the case $\alpha = 1/\sqrt{3}$, for the function (13) at the *critical points* $z = \pm i r_c(1/\sqrt{3})$, and for the functions (14) and (15) at the *critical points* $z = r_c(1/\sqrt{3})$ and $z = -r_c(1/\sqrt{3})$, respectively;

3) in the case $1/\sqrt{3} < \alpha < 1$, for the functions (14) and (15) at the *critical points* $z = r_c(\alpha)$ and $z = -r_c(\alpha)$, respectively.

Proof. From the results in Robertson (1936) and Reade & Todorov (1981) it follows that, if $\varphi \in N_2$, then the function

$$\psi(z) \equiv z\varphi'(z) \tag{51}$$

is typically real for $|z| < 1$. Now Theorem 1 from Todorov (1983a), applied to the radius of starlikeness of order alpha of the set of functions (51), yields the assertions in Theorem 3 concerning the radius of convexity of order alpha of the set N_2 of functions

$$\varphi(z) \equiv \int_0^z \frac{\psi(\zeta)}{\zeta}\, d\zeta, \quad |z| < 1. \tag{52}$$

This completes the proof of Theorem 3.

If we replace z by $1/z$, then Theorem 3 yields the following result for the class N_1.

Theorem 4. The radius of convexity $R_c(\alpha)$ of order $-\alpha$, $0 \leqslant \alpha < 1$, of the class N_1 is

$$R_c(\alpha) \equiv \sqrt{\frac{\sqrt{\alpha^2+8} - \alpha + 2}{\sqrt{\alpha^2+8} - \alpha - 2}}, \quad 0 \leqslant \alpha < \frac{1}{\sqrt{3}}, \tag{53}$$

$$R_c\left(\frac{1}{\sqrt{3}}\right) \equiv 2 + \sqrt{3}, \quad \alpha = \frac{1}{\sqrt{3}}, \tag{54}$$

$$R_c(\alpha) \equiv \frac{1+\alpha}{1-\alpha}, \quad \frac{1}{\sqrt{3}} < \alpha < 1, \tag{55}$$

i.e. for each function $f \in N_1$, in the disc $|z| \geqslant R_c(\alpha)$, the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \leqslant -\alpha, \quad 0 \leqslant \alpha < 1, \tag{56}$$

holds, where the equality holds only for the following extremal functions:

1) in the case $0 \leqslant \alpha < 1/\sqrt{3}$, for the function (44) at the *critical points* $z = \pm i R_c(\alpha)$;

2) in the case $\alpha = 1/\sqrt{3}$, for the function (44) at the *critical points* $z = \pm i R_c(1/\sqrt{3})$,

and for the functions (45) and (46) at the *critical points* $z = R_c(1/\sqrt{3})$ and $z = -R_c(1/\sqrt{3})$, respectively;

3) in the case $1/\sqrt{3} < \alpha < 1$, for the functions (45) and (46) at the *critical points* $z = R_c(\alpha)$ and $z = -R_c(\alpha)$, respectively.

In particular, for $\alpha = 0$, Theorems 3 and 4 yield the corresponding results in Todorov (1980) and Reade & Todorov (1981).

Remark. In particular, if $\mu(t)$ in (1) and (2) is a step-function with n jumps, then the classes N_1 and N_2 are reduced to the corresponding classes $R_1(I)$ and $R_2(I)$ of rational functions

$$f(z) \equiv \sum_{k=1}^{n} \frac{A_k}{z - a_k}, \tag{57}$$

and

$$\varphi(z) \equiv f\left(\frac{1}{z}\right) \equiv \sum_{k=1}^{n} \frac{A_k z}{1 - a_k z}, \tag{58}$$

where

$$\sum_{k=1}^{n} A_k = 1, \quad A_k > 0, \ -1 \leqslant a_k \leqslant a_{k+1} \leqslant 1, \ 1 \leqslant k \leqslant n-1, \ n \geqslant 2. \tag{59}$$

For the separate classes $R_1(I)$ and $R_2(I)$ the corresponding Theorems 1, 2, 3 and 4 hold as well (see also Todorov (1983b)).

REFERENCES

Reade, M.O. & Todorov, P.G. 1981. The radii of starlikeness and convexity of certain Nevanlinna analytic functions. Proceedings of the American Mathematical Society **83**: 289–95.

Robertson, M.S. 1936. Analytic functions starlike in one direction. American Journal of Mathematics **58**: 465–72.

Todorov, P.G. 1980. Limite de convexité d'une classe de Nevanlinna de fonctions analytiques. Archiv der Mathematik **34**: 127–31.

Todorov, P.G. 1983a. The radii of starlikeness and convexity of order alpha of typically real functions. Annales Academiae Scientiarum Fennicae Series A. 1. Mathematica **8**: 93–106.

Todorov, P.G. 1983b. The radii of starlikeness of certain rational functions with real simple poles. Resultate der Mathematik **6**: 95–99.

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أنصاف الأقطار لمثيل النجمة والانحداب من الرتبة ألفا لبعض دوال نيڤانلينا التحليلية

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خلاصة

تمكن الباحث من تعيين أنصاف الأقطار للدوال التحليلية مثيلة النجمة ذات الرتبة الفا والدوال التحليلية المحدبة ذات الرتبة الفا ، وذلك من النوع $\int_{-1}^{1} d\mu(t)/(z-t)$ والنوع $\int_{-1}^{1} z\,d\mu(t)/(1-tz)$ حيث $\mu(t)$ يمثل قياس الاحتمال على الفترة [1,1−] .