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Commutativity of some A-convex algebras

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ABSTRACT

Using structure theorems, some of which are ours, we extend to locally A-convex algebras results on commutativity in Banach algebras in the spirit of Hirschfield & Żelazko and Le Page.

INTRODUCTION

Le Page (1967) and Hirschfield & Żelazko (1968) began the characterization of commutative Banach algebras. Since then no attempt has been made to extend the results to topological non-Banach algebras. The aim of this paper is to realize this extension for a type of algebras occurring in functional analysis. We, in particular, obtain theorems of Gelfand–Mazur type.

Complete locally m-convex algebras (l.m.c.a) are projective limits of Banach algebras (Michaël 1952). This allows us to obtain some theorems of Le Page (1967) and Hirschfield & Żelazko (1968) type.

Complete A-convex algebras (l.A-c.a) are projective limits of A-normed algebras (Cochran 1973). This fact is not adequate for the questions we are interested in. But Theorem 1 of Oudadess (1982) provides any unitary l.A-c.a with locally m-convex topology while Theorem III.2 of Oudadess (1985) provides unitary l.u.A-c.a with an algebra norm. These facts make it possible to establish the commutativity of some locally A-convex algebras (see Sections 2 and 3). It is worth noting that some of the conditions we are considering here were, for different reasons, introduced too in some of the works on topological algebras (Morris & Wulbert 1967) where no commutativity problem is involved.

PRELIMINARIES

Let (E, τ) be a locally convex space, and $(P_\lambda)_\lambda$ a family of semi-norms defining its topology. If E is endowed with an algebra structure such that the multiplication is separately continuous, we say that (E, τ) is a locally convex algebra (l.c.a). A l.c.a is said to be A-convex (l.A-c.a) if for each λ and each x, there exists $M(\lambda, x) > 0$ and $N(\lambda, x) > 0$ such that

$$P_\lambda(xy) \leqslant M(\lambda, x) P_\lambda(y),$$

and

$$P_\lambda(yx) \leqslant N(\lambda, x) P_\lambda(y) \quad \text{for every } y.$$

We say that a l.A-c.a is uniformly A-convex (l.u.A-c.a) if $M(\lambda, x) = M(x)$ and $N(\lambda, x) = N(x)$ are independent of λ. A l.A-c.a is said to be m-convex (l.m.c.a) if

$$M(\lambda, x) = P_\lambda(x) \quad \text{and} \quad N(\lambda, x) = P_\lambda(x).$$

Example 1. Let $C_b(\mathbb{R})$ be the algebra of complex continuous and bounded functions on $\mathbb{R}$, endowed with the topology given by the family $(P_\Phi)_\Phi$ of semi-norms where $P_\Phi(f) = \mathrm{Sup}'\{|f(x)\Phi(x)| : x \in \mathbb{R}\}$ and Φ runs over the ideal $C_b^0(\mathbb{R})$ of elements of $C_b(\mathbb{R})$ which tends to zero at infinity. This is a l.u.A-c.a which is not a l.m.c one (see Cocharn *et al.* 1970).

Example 2. Let $C(\mathbb{R})$ be the algebra of complex continuous function on $\mathbb{R}$ endowed with the topology of uniform convergence on compacts. This is a l.m.c.a which is not a l.u.A-c one.

Example 3. By considering the product $C_b(\mathbb{R}) \times C(\mathbb{R})$ endowed with the product topology we get a l.A-c.a which is not a l.u.A-c.a.

For other examples and properties of these algebras, see Akkar (1975, 1976), Cochran (1973), Oudadess (1983a, b, 1985) and Żelazko (1971).

In the sequel $\mathbb{C}$ will denote the field of complex numbers.

1. LOCALLY *m*-CONVEX CASE

We first recall a structure theorem (Michaël 1952) which will frequently be of use in this paper. Let (E, τ) be a complex complete l.m.c.a with $(P_\lambda)_{\lambda \in \Lambda}$ (where Λ is a directed set of indices) a family of submultiplicative semi-norms defining the topology τ. Then (E, τ) is the projective limit (algebraic and topologic) of the Banach algebras E_λ where E_λ is the completion of the quotient of E, endowed with the semi-norm P_λ alone, by the closed ideal $N_\lambda = \{x \in E / P_\lambda(x) = 0\}$. An element x in E is written $(\pi_\lambda(x))_\lambda$ where π_λ is the quotient map of E on E_λ. The morphisms for the projective limit are the $\Pi_{\mu\lambda}$ $(\lambda \leqslant \mu)$ defined by

$$\Pi_{\mu\lambda}(\pi_\mu(x)) = \pi_\lambda(x).$$

We begin with some theorems of the Gelfand–Mazur type.

Theorem 1.1. Let (E, τ) be a complex complete l.A-c.a such that

$$P_\lambda(xy) = P_\lambda(x) \cdot P_\lambda(y), \quad \forall x, \quad \forall y \tag{1}$$

Then E is the diagonal of a product whose factors are isomorphic to $\mathbb{C}$.

Proof. The algebra is of course l.m.c. Then it is a projective limit of the Banach algebras E_λ. By the construction of the E_λ, they also satisfy (1). Hence, for any λ, E_λ is isomorphic to $\mathbb{C}$ (Belfi & Doran 1977). But a projective limit whose factors are equal and the relative morphisms all reduce to the identity map is exactly the diagonal of the product of its factors by Example 2 on page 77 (Bourbaki 1967).

By the same arguments the hypothesis of the previous theorem can be weakened and we get Theorem 1.2.

Theorem 1.2. Let (E, τ) be a complex complete l.m.c.a such that

$$\forall \lambda, \exists \alpha_\lambda > 0 : P_\lambda(xy) \geqslant \alpha_\lambda P_\lambda(x) \cdot P_\lambda(y), \forall x, \forall y \tag{2}$$

Then E is the diagonal of a product whose factors are isomorphic to $\mathbb{C}$.

We now deal with the commutativity problem. It is clear that the conditions we will consider are inherited by the factors of which the algebra is the projective limit. Then using Theorem 5.1 and Theorem 5.8 of Belfi & Doran (1977) we have Theorem 1.3.

Theorem 1.3. Let (E, τ) be a complex unitary and complete l.m.c.a satisfying one of the following conditions:

$$\forall^\lambda, \exists \alpha_\lambda > 0 : [P_\lambda(x)]^2 \leqslant \alpha_\lambda P_\lambda(x^2), \forall x \tag{3}$$

$$\forall \lambda, \exists \alpha_\lambda > 0 : P_\lambda(xy) \leqslant \alpha_\lambda P_\lambda(yx); \forall x, y \tag{4}$$

Then E is commutative.

The following result describes a situation where an element is forced to be in the center of the algebra, i.e. commuting with any other element. Arguing in the same way and using Theorem II.6 of Oudadess (1983c) we obtain Theorem 1.4.

Theorem 1.4. Let (E, τ) be a complex complete l.m.c.a and let c be an element of E such that

$$\forall \lambda, P_\lambda(cx + \gamma x) \leqslant P_\lambda(xc + \gamma x), \forall x, \forall \gamma \in \mathbb{C} \tag{5}$$

Then c is the centre of E.

2. LOCALLY A-CONVEX CASE

In this section, it is in the locally uniformly A-convex case that we obtain results analogous to those on Banach algebras. In the locally A-convex case (not l.m.c nor u.A-c) we have to put much stronger conditions.

We first give a strong propertty of l.u.A-c.a.

Theorem 2.1. Let (E, τ) be a complex unitary and sequentially complete l.u.A-c.a whose topology is given by a family (P_λ) of semi-norms. Then $\|x\|_0 = \mathrm{Sup}\{P_\lambda(x) : \lambda \in \Lambda\}$ makes of E a banach algebra.

Proof. Theorem III.2 of Oudadess (1985) provides such algebra with a Banach algebra norm. If we examine the construction of this norm it appears that it is equivalent to $\|\cdot\|_0$.

For the proofs of the following results one has to note that the conditions considered are inherited by the factors E_λ and to apply the same results on Banach algebras (cf. Theorem 4.1, Corollary 4.11, Corollary 4.14, Theorem 5.1, Corollary 5.2, and Theorem 5.8, of Belfi & Doran (1977)).

Theorem 2.2. Let (E, τ) be a complex unitary and sequentially complete l.u.A-c.a satisfying one of the following conditions:

$$P_\lambda(xy) = P_\lambda(x) \cdot P_\lambda(y), \forall y, \forall x, y \tag{6}$$

$$\exists \alpha > 0 : P_\lambda(xy) \geqslant \alpha P_\lambda(x) \cdot P_\lambda(y), \forall \lambda, \forall x, y \tag{7}$$

$$\exists \alpha > 0 : P_\lambda(x) \cdot P_\lambda(x^{-1}) \leqslant \alpha P_\lambda(xx^{-1}), \forall x \text{ invertible} \tag{8}$$

Then E is commutative.

Remark 2.3. Theorem 2.2 is not valid in the real case. For example, the algebra of quaternions satisfies (6) and is not commutative.

Remark 2.4. The condition $P_\lambda(x^{-1}) = [P_\lambda(x)]^{-1}$ does not seem to be inherited by $\|\cdot\|_0$ of Theorem 2.1.

Remark 2.5. In the normed case we do not need the completeness when considering (6) and (7). We know that there is on any unitary Hausdorff l.u.A-c.a an algebra norm $\| \ \|_c$ (see Cochran 1973, p. 477) given by $\|x\| = \sup_\lambda \sup\{p_\lambda(x \cdot y) : p_\lambda(y) \leqslant 1\}$ but it does not seem that (6) and (7) are inherited by this norm.

Theorem 2.6. Let (E, τ) be a complex unitary and sequentially complete l.u.A-c.a satisfying one of the following conditions:

$$\exists \alpha > 0 : [P_\lambda(x)]^2 \leqslant \alpha P_\lambda(x^2), \forall \lambda, \forall x \tag{9}$$

$$\exists \alpha > 0 : P_\lambda(xy) \leqslant \alpha P_\lambda(yx), \forall \lambda, \forall x, y \tag{10}$$

$$\exists \alpha > 0 : P_\lambda(x) \leqslant \alpha \rho(x), \forall \lambda, \forall y \tag{11}$$

Then E is commutative.

Remark 2.7. Condition (11) makes sense since, by Theorem 2.1, $\rho(x)$ is finite for every x, where $\rho(x)$ is the spectral radius of x.

Remark 2.8. If we try to weaken (9) and (10) by allowing α to depend on λ, it may happen that $\operatorname{Sup} \alpha_\lambda = +\infty$. However we can weaken the hypothesis even in the locally A-convex case if we demand

$$P_\lambda(yux) \leqslant \alpha_\lambda P_\lambda(yxu), \forall y, u, x$$

or at least for every u such that $P_\lambda(u) \leqslant 1$. This is a reasonable condition since it is satisfied by commutative algebras.

Remark 2.9. In the normed case, condition (11) implies (9). But it does not seem to be the case in general.

Using Theorem 5.9 of Belfi & Doran (1977) and the norm $\|\cdot\|_0$ of Theorem 1.2 we get Theorem 2.9.

Theorem 2.10. Let (E, τ) be a complex unitary and sequentially complete l.u.A-c.a and c an element of E verifying

$$\forall \lambda; P_\lambda(cx + \gamma x) \leqslant P_\lambda(xc + \gamma x), \forall x, \forall \gamma \in \mathbb{C} \tag{12}$$

Then c is in the center of E.

3. ANOTHER CASE OF COMMUTATIVITY

We now consider an algebraic condition. If E is an algebra and x is an element of E, we put $E \cdot x = \{yx : y \in E\}$ and $Ex^2 = \{yx^2 : y \in E\}$.

Theorem 3.1. Let (E, τ) be a complex unitary and complete l.A-c.a such that

$$Ex^2 = Ex, \forall x \tag{13}$$

Then E is commutative and semi-simple.

Proof. The condition (13) is algebraic and hence is valid for the l.m.c.a $(E, M(\tau))$ of Theorem 1 in Oudadess (1982). If we consider the Banach algebras E_λ of which $(E, M(\tau))$ is the projective limit, they also verify (13). But then they are commutative (see Belfi & Doran 1977). Hence E is commutative. It is also semi-simple because each E_λ is: E_λ is semi-simple by the theorem just mentioned and E is then semi-simple (see Żelazko 1971).

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مسائل ابدالية في بعض جبريات أ ـ محدبة

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خلاصة

باستخدام مبرهنات بنيوية ، بعضها يرجع للمؤلف ، أمكن تعميم نتائج حول الابدالية في جبريات باناخ الى جبريات أ ـ محدبة محليا ، وذلك وفق توجهات هيرشفيلد وزلازكو وليپاج .