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Bending of a thin circular plate uniformly loaded over a concentric ellipse and symmetrically supported at four points on its axes

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ABSTRACT

Within the restrictions of the classical small bending theory of thin plates, exact expressions are obtained in series form for the deflection at any point of a thin isotropic circular plate uniformly loaded over a concentric ellipse and supported at the four vertices of a concentric rhombus whose diagonals coincide with the axes of the ellipse. Formulae for the moments and shears are given and numerical results are presented in tabular and graphical forms.

1. INTRODUCTION

The transverse flexure of thin isotropic circular plates subject to various distributions of normal loadings and supported in different ways has been considered by many authors. For loadings spread over an eccentric circle and supports at interior or boundary points, the problems were solved by one of the authors (Bassali 1957, 1958). Solutions corresponding to the same loadings and supports along a circle concentric with the plate were also found by Bassali & Barsoum (1966a, b). Elastically restrained circular plates under uniform or uniformly varying loadings over a concentric ellipse were analysed by Bassali & Nassif (1959) and Bassali (1959). The method of complex potentials was applied by Nassif (1966) to deal with the case of uniform loading over a concentric ellipse and support along a concentric circle. The two cases of uniform or hydrostatic pressure over a concentric ellipse and four supports at the corners of a concentric rectangle whose sides are parallel to the axes of the ellipse were treated in two recent papers by Bassali (1986a, b). Symmetrically loaded thin circular plates supported at equally spaced points on a concentric circle have been studied by Kirstein *et al.* (1966), and Kirstein & Woolley (1967, 1968), and there is good agreement between their experimental results and the theoretical results of Bassali (1957). Thin circular plates on multi-point supports were also analysed by Yu & Pan (1966), Leissa & Wells (1966), Vaughan (1970), Chantaramungkorn *et al.* (1973) and Williams & Brinson (1974). In two recent papers one of the authors (Bassali 1984, 1986c) obtained the deflection surface of a thin circular annular plate supported at equispaced points on a concentric circle and symmetrically loaded either over its entire surface or over a concentric circular annulus. The authors (1988) investigated the case of a thin circular plate under normal paraboloidal loading over a concentric ellipse and supported at the four corners of a concentric rectangle whose sides are parallel to the axes of the ellipse. In this paper we consider the case of uniform pressure over the area of a concentric ellipse and four supports at the vertices of a concentric rhombus whose diagonals coincide with the axes of the ellipse. Numerical values of deflections, moments and shears are tabulated and graphs showing their variation along various radii and concentric circles are plotted.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let C denote the boundary of a thin elastic circular plate of uniform small thickness $2h$, radius c, flexural rigidity D and made of isotropic homogeneous material with Poisson's ratio v. Let $z=x+iy=re^{i\theta}$ be the complex variable of any point N in the mid-plane of the plate referred to its centre O. We assume that (i) the plate is subject to uniform normal loading of intensity p distributed over the area of the ellipse $x^2/a^2+y^2/b^2=1$ $(0\leqslant b\leqslant a\leqslant c)$, (ii) the plate is supported at the four points $P_1(s_1,0)$, $P_1'(-s_1,0)$, $P_2(0,s_2)$ and $P_2'(0,-s_2)$, (iii) the boundary C is free (see Fig. 1). Let the indices 1, 2, refer to the region inside the boundary Γ of the ellipse and that between Γ and C, respectively. If N_1, N_1, N_2, N_2 are the reactions at the four points of support then $2N_1+2N_2=L$, where $L=\pi abp$ is the total load on the plate. Assuming that $N_1=\lambda_1 L$, $N_2=\lambda_2 L$ we have

$$\lambda_1+\lambda_2=\tfrac{1}{2}. \tag{1}$$

It is worthy of mention here that in the problem treated by Bassali (1986a) the four points of support lie at

$$z=\pm s_1\pm is_2=se^{i\gamma_\lambda}\quad(\lambda=1,2,3,4,\ \gamma_1=\gamma,\ \gamma_2=\pi-\gamma,\ \gamma_3=\gamma-\pi,\ \gamma_4=-\gamma),$$

and the reactions are statically determinate from the conditions of equilibrium, each being equal to $L/4$. In the problem under consideration the reactions are statically indeterminate and in order to find them the elastostatic problem of bending of the plate must be completely solved. It will be seen that the values of λ_1 and λ_2 depend on the positions of the supports. We assume that w is the small deflection at N, measured positively downwards, and that it vanishes at the four points of support. Symmetry shows that N can be taken in the positive quadrant of the plate. If w_1, w_2 are the deflections in the two regions then

$$\nabla^4 w_1=p/D,\quad \nabla^4 w_2=0, \tag{2}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{4\,\partial^2}{\partial z\,\partial\bar{z}}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{3}$$

The deflection w_2 must satisfy the following conditions for the boundary C to be free (Bassali 1986a, b):

$$[(d^2+vr^{-1}d+vr^{-2}d'^2)w_2]_{r=c}=0, \tag{4a}$$

$$[\{d^3+r^{-1}d^2-r^{-2}d+(2-v)r^{-2}dd'^2+(v-3)r^{-3}d'^2\}w_2]_{r=c}=0, \tag{4b}$$

where $d=\partial/\partial r$, $d'=\partial/\partial\theta$. The deflections w_1, w_2 must conform to the following transition conditions along Γ:

$$[w]_1^2 = [\partial w/\partial z]_1^2 = [\partial^2 w/\partial z\,\partial\bar{z}]_1^2 = [\partial^3 w/\partial z^2\,\partial\bar{z}]_1^2 = 0. \tag{5}$$

Furthermore, the deflection w must exhibit the appropriate singular behaviour at the four points of support.

3. METHOD AND SOLUTION

The general solutions of (2) are

$$w_j = 2\,\mathrm{Re}[\bar{z}\Omega_j(z) + \omega_j(z)] + W_j(z, \bar{z}) \quad (j = 1, 2), \tag{6}$$

where

$$W_1(z, \bar{z}) = pz^2\bar{z}^2/64D, \quad W_2(z, \bar{z}) = 0, \tag{7}$$

and $\Omega_j(z)$, $\omega_j(z)$ are four complex potentials to be determined. It was shown by Bassali & Nassif (1959, p. 105) that the continuity requirements (5) along Γ are satisfied by

$$2k[\Omega]_1^2 = \frac{d^2z^3}{6abf^2} - \frac{1}{3}\left(2 + \frac{z^2}{f^2}\right)Z + z\ln\frac{z+Z}{a+b}, \tag{8}$$

$$2k[\omega]_1^2 = ab\left(\frac{1}{4} + \frac{z^2}{f^2}\right) - \frac{1}{3}\left(\frac{1}{8ab} + \frac{ab}{f^4}\right)z^4 + \frac{d^2}{4}\left\{\frac{zZ}{3f^2}\left(\frac{2z^2}{f^2} - 5\right) + \ln\frac{z+Z}{a+b}\right\}, \tag{9}$$

where

$$k = 8\pi D/L, \quad d^2 = a^2 + b^2, \quad f^2 = a^2 - b^2, \quad Z = \sqrt{z^2 - f^2}, \tag{10}$$

and the branch of Z chosen is that which is positive when z is real and $z^2 > f^2$. The singular parts of the complex potentials near the points of application of concentrated forces are given by Eqns (2.18), p. 732 of Bassali (1957). Making use of these equations and applying (8), (9) we assume that

$$2k\Omega_1 = \sum_0^\infty C_n z^{2n+1} - \frac{d^2z^3}{6abf^2} - \lambda_1\left(Z_1\ln\frac{Z_1}{c} + Z_1'\ln\frac{Z_1'}{c}\right) - \lambda_2\left(Z_2\ln\frac{Z_2}{c} + Z_2'\ln\frac{Z_2'}{c}\right), \tag{11a}$$

$$2k\omega_1 = \sum_0^\infty A_n z^{2n} - ab\left(\frac{1}{4} + \frac{z^2}{f^2}\right) + \frac{1}{3}\left(\frac{1}{8ab} + \frac{ab}{f^4}\right)z^4 + \lambda_1\left(\bar{z}_1Z_1\ln\frac{Z_1}{c} + \bar{z}_1'Z_1'\ln\frac{Z_1'}{c}\right)$$
$$+ \lambda_2\left(\bar{z}_2Z_2\ln\frac{Z_2}{c} + \bar{z}_2'Z_2'\ln\frac{Z_2'}{c}\right); \tag{11b}$$

$$2k\Omega_2 = \sum_0^\infty C_n z^{2n+1} + z\ln\frac{z+Z}{a+b} - \frac{1}{3}\left(2 + \frac{z^2}{f^2}\right)Z - \lambda_1\left(Z_1\ln\frac{Z_1}{c} + Z_1'\ln\frac{Z_1'}{c}\right)$$
$$- \lambda_2\left(Z_2\ln\frac{Z_2}{c} + Z_2'\ln\frac{Z_2'}{c}\right), \tag{12a}$$

$$2k\omega_2 = \sum_0^\infty A_n z^{2n} + \frac{d^2}{4}\left\{\frac{zZ}{3f^2}\left(\frac{2z^2}{f^2} - 5\right) + \ln\frac{z+Z}{a+b}\right\}$$
$$+ \lambda_1\left(\bar{z}_1Z_1\ln\frac{Z_1}{c} + \bar{z}_1'Z_1'\ln\frac{Z_1'}{c}\right) + \lambda_2\left(\bar{z}_2Z_2\ln\frac{Z_2}{c} + \bar{z}_2'Z_2'\ln\frac{Z_2'}{c}\right), \tag{12b}$$

where, for uniformity reasons, the terms involving Z appear in Ω_2, ω_2, $z_1 = s_1$,

$z'_1 = -s_1$, $z_2 = is_2$, $z'_2 = -is_2$, $Z_j = z - z_j$, $Z'_j = z - z'_j$ $(j = 1, 2)$, and A_n, C_n $(n = 0, 1, 2, \ldots)$ are real constants which, together with λ_1, λ_2, will be determined from the boundary conditions (4a, b), the two conditions that the deflection vanishes at the support points and Eqn (1). Substitution from (11a, b), (12a, b) and (7) in (6) yields

$$kw_1 = \sum_0^{\infty} (A_n + C_n r^2) r^{2n} \cos 2n\theta - \lambda_1 S_1 - \lambda_2 S_2 - ab\left(\frac{1}{4} + \frac{r^2}{f^2}\cos 2\theta\right)$$
$$+ \frac{r^4}{24ab}\left\{3 - \frac{4d^2}{f^2}\cos 2\theta + \left(1 + \frac{8a^2b^2}{f^4}\right)\cos 4\theta\right\}, \tag{13a}$$

$$kw_2 = \sum_0^{\infty} (A_n + C_n r^2) r^{2n} \cos 2n\theta - \lambda_1 S_1 - \lambda_2 S_2 + (r^2 + \tfrac{1}{4}d^2) \ln \frac{|z + Z|}{a + b}$$
$$+ \tfrac{1}{12} Re\left\{\frac{2d^2 z^4}{f^4} - (4r^2 + 5d^2)\frac{z^2}{f^2} - 8r^2\right\}\sqrt{1 - \frac{f^2}{z^2}}, \tag{13b}$$

where

$$S_1 = R_1^2 \ln \frac{R_1}{c} + R_1'^2 \ln \frac{R_1'}{c}, \quad S_2 = R_2^2 \ln \frac{R_2}{c} + R_2'^2 \ln \frac{R_2'}{c}, \tag{14}$$

$$\left.\begin{aligned} R_1^2 &= r^2 + s_1^2 - 2s_1 r \cos\theta, & R_1'^2 &= r^2 + s^2 + 2s_1 r \cos\theta \\ R_2^2 &= r^2 + s_2^2 - 2s_2 r \sin\theta, & R_2'^2 &= r^2 + s_2^2 + 2s_2 r \sin\theta. \end{aligned}\right\} \tag{15}$$

Applying Eqns (4.28a), (4.29a) and (4.31a) (with $\gamma_\lambda = 0, \pi, \pi/2, -\pi/2$) of Bassali (1986a, p. 166) we obtain the following expansion for kw_2:

$$kw_2 = A'_0 + B'_0 \ln \frac{r}{c} + C'_0 r^2 + \sum_1^{\infty} (A'_n r^{2n} + B'_n r^{-2n} + C'_n r^{2+2n} + D'_n r^{2-2n}) \cos 2n\theta, \tag{16}$$

where $r \geqslant$ the largest of f, s_1, s_2 and

$$A'_0 = A_0 - 2\lambda_1 s_1^2 - 2\lambda_2 s_2^2 + \tfrac{1}{4}d^2\left(\tfrac{3}{4} - \ln \frac{a + b}{2c}\right), \tag{17a}$$

$$B'_0 = \tfrac{1}{4}d^2 - 2\lambda_1 s_1^2 - 2\lambda_2 s_2^2, \quad C'_0 = C_0 - \tfrac{1}{2} - \ln \frac{a + b}{2c}, \tag{17b}$$

$$A'_1 = A_1 - \frac{d^2}{2f^2}, \quad A'_2 = A_2 + \frac{d^2}{6f^4}, \quad C'_1 = C_1 - \frac{1}{3f^2}, \tag{17c}$$

$$A'_n = A_n (n \geqslant 3), \quad B'_n = \frac{\lambda_1 s_1^{2n+2} + (-1)^n \lambda_2 s_2^{2n+2}}{n(2n + 1)} - \frac{d^2 f^{2n} \delta_n}{4n(n + 1)(n + 2)} \; (n = 1, 2, \ldots), \tag{17d}$$

$$C'_n = C_n (n \geqslant 2), \quad D'_n = \frac{f^{2n}\delta_n}{2n(n + 1)(2n - 1)} - \frac{\lambda_1 s_1^{2n} + (-1)^n \lambda_2 s_2^{2n}}{n(2n - 1)} \; (n = 1, 2, \ldots), \tag{17e}$$

$$\delta_n = 2^{-2n}\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)}. \tag{18}$$

Introducing (16) in (4a, b), equating the coefficients of $\cos 2n\theta$ ($n = 0, 1, 2, \ldots$) in the resulting identities to zero and solving the linear systems of equations obtained we have

$$C_0' = \beta(\tfrac{1}{8}v^2 - \lambda_1 u_1^2 - \lambda_2 u_2^2), \tag{19}$$

$$C_0 = \tfrac{1}{2} + \ln\frac{t_1 + t_2}{2} + \beta(\tfrac{1}{8}v^2 - \lambda_1 u_1^2 - \lambda_2 u_2^2), \tag{20}$$

$$A_n' = \frac{c^{2-2n}}{n\kappa}\left[\lambda_1 u_1^{2n}(u_1^2 - \alpha_n) + (-1)^n \lambda_2 u_2^{2n}(u_2^2 - \alpha_n) + \frac{t^{2n}\delta_n}{4(n+1)}\left(2\alpha_n - \frac{2n+1}{n+2}v^2\right)\right], \quad (n \geqslant 1), \tag{21}$$

$$C_n' = \frac{c^{-2n}}{\kappa}\left[\lambda_1 u_1^{2n}\left(\frac{1}{n} - \frac{2u_1^2}{2n+1}\right) + (-1)^n \lambda_2 u_2^{2n}\left(\frac{1}{n} - \frac{2u_2^2}{2n+1}\right) + \frac{t^{2n}\delta_n}{2(n+1)}\left(\frac{v^2}{n+2} - \frac{1}{n}\right)\right], \quad (n \geqslant 1), \tag{22}$$

where

$$v = d/c, \quad u_1 = s_1/c, \quad u_2 = s_2/c, \quad t_1 = a/c, \quad t_2 = b/c, \quad t = f/c, \tag{23}$$

$$\beta = (1-\nu)/(1+\nu), \quad \kappa = (3+\nu)/(1-\nu), \quad \alpha_n = \frac{\kappa^2 + 4n^2 - 1}{2n(2n-1)}. \tag{24}$$

Substituting for A_1, A_2 and C_1 from (17c) in (13a, b) we get

$$kw_1 = \tfrac{1}{2}m\left(1 - \frac{r^2}{3ab}\right)r^2\cos 2\theta + \frac{r^4}{8ab}(1 + \tfrac{1}{3}m^2\cos 4\theta) - \tfrac{1}{4}ab - \lambda_1 S_1 - \lambda_2 S_2 + A_0 + C_0 r^2 + \sum_1^\infty (A_n' + C_n' r^2) r^{2n}\cos 2n\theta, \tag{25a}$$

$$kw_2 = (\tfrac{1}{2}d^2 + \tfrac{1}{3}r^2)\frac{r^2}{f^2}\cos 2\theta - \frac{d^2 r^4}{6f^4}\cos 4\theta + (r^2 + \tfrac{1}{4}d^2)\ln\frac{|z+Z|}{a+b} - \lambda_1 S_1 - \lambda_2 S_2 + \tfrac{1}{12}\,\mathrm{Re}\left\{\frac{2d^2 z^4}{f^4} - (4r^2 + 5d^2)\frac{z^2}{f^2} - 8r^2\right\}\sqrt{1 - \frac{f^2}{z^2}} + A_0 + C_0 r^2 + \sum_1^\infty (A_n' + C_n') r^{2n}\cos 2n\theta, \tag{25b}$$

where

$$m = (a-b)/(a+b), \tag{26}$$

and the values of the other constants except A_0 are furnished by Eqns (19)–(22). Putting $b = a$, $m = 0$ in (25a) and finding the limit of (25b) as $f \to 0$ we arrive at the equations of the deflection surface of the plate corresponding to uniform loading over a concentric circle and four symmetrical supports on two perpendicular diameters. There remains now the determination of A_0, λ_1 and λ_2 from Eqn (1) and the conditions $w_{P_1} = w_{P_2} = 0$. Assuming that both P_1 and P_2 lie in the unloaded region and setting $z = s_1$, $z = is_2$ in (13b) we get the two following equations which are linear in λ_1 and λ_2:

$$\frac{A_0}{s_1^2} + C_0 + \sum_1^\infty \left(\frac{A_n}{s_1^2} + C_n\right) s_1^{2n} - 4\lambda_1 \ln(2u_1) - \lambda_2 \left(1 + \frac{u_2^2}{u_1^2}\right) \ln(u_1^2 + u_2^2)$$

$$+ \left(1 + \frac{v^2}{4u_1^2}\right) \ln \frac{u_1 + \sqrt{u_1^2 - t^2}}{t_1 + t_2} + \frac{1}{6}\left(\frac{v^2 u_1^2}{t^4} - \frac{4u_1^2 + 5v^2}{2t^2} - 4\right)\sqrt{1 - \frac{t^2}{u_1^2}} = 0, \quad (27a)$$

$$\frac{A_0}{s_2^2} + C_0 + \sum_1^\infty (-1)^n \left(\frac{A_n}{s_2^2} + C_n\right) s_2^{2n} - \lambda_1 \left(1 + \frac{u_1^2}{u_2^2}\right) \ln(u_1^2 + u_2^2) - 4\lambda_2 \ln(2u_2)$$

$$+ \left(1 + \frac{v^2}{4u_2^2}\right) \ln \frac{u_2 + \sqrt{u_2^2 + t^2}}{t_1 + t_2} + \frac{1}{6}\left(\frac{v^2 u_2^2}{t^4} + \frac{4u_2^2 + 5v^2}{2t^2} - 4\right)\sqrt{1 + \frac{t^2}{u_2^2}} = 0. \quad (27b)$$

If all the support points lie in the loaded region, then Eqn (13a) can be used to write down the appropriate conditions. Similarly we deal with the cases in which two point supports lie in the loaded region and the other two supports lie in the unloaded region.

The deflection w_0 at the centre O of the plate is given by

$$w_0 = \frac{L}{8\pi D}(A_0 - \tfrac{1}{4}ab - 2\lambda_1 s_1^2 \ln u_1 - 2\lambda_2 s_2^2 \ln u_2). \quad (28)$$

4. MOMENTS AND SHEARS

Introducing (7), (11a, b) and (12a, b) in the standard formulae given by Eqns (2.7a, b) and (2.8) of Bassali (1986a, p. 163) we get, after extensive algebraic manipulation, the following expressions for the bending and twisting moments and shearing forces at any point of the plate:

$$\begin{matrix} M_r^1 \\ M_\theta^1 \end{matrix} = \frac{(1+\nu)L}{4\pi}\Bigg[(1 \pm \tfrac{1}{2}\beta)\left(1 - \frac{r^2}{2ab}\right) + \lambda_1 \ln V_1 + \lambda_2 \ln V_2 \mp \beta\left(\frac{1}{4ab} + \frac{2ab}{f^4}\right) r^2 \cos 4\theta$$

$$+ \left\{\frac{1 \pm \beta}{2ab} d^2 r^2 \pm \beta ab\right\} \frac{\cos 2\theta}{f^2} \mp 2\beta \left\{\frac{\lambda_1 u_1^2}{V_1^2}(\rho^2 + u_1^2)\sin^2\theta + \frac{\lambda_2 u_2^2}{V_2^2}(\rho^2 + u_2^2)\cos^2\theta\right\}$$

$$- C_0 - \sum_1^\infty \left\{(2n+1)(1 \pm n\beta) C_n \pm \frac{\beta n(2n-1)}{r^2} A_n\right\} r^{2n} \cos 2n\theta\Bigg], \quad (29a, b)$$

$$M_{r\theta}^1 = \frac{(1-\nu)L}{4\pi}\Bigg[\left\{\frac{\lambda_1 u_1^2}{V_1^2}(u_1^2 - \rho^2) - \frac{\lambda_2 u_2^2}{V_2^2}(u_2^2 - \rho^2) + \frac{1}{f^2}\left(ab + \frac{d^2 r^2}{2ab}\right)\right\} \sin 2\theta$$

$$- \left(\frac{1}{4ab} + \frac{2ab}{f^4}\right) r^2 \sin 4\theta - \sum_1^\infty n\left\{(2n+1) C_n + \frac{2n-1}{r^2} A_n\right\} r^{2n} \sin 2n\theta\Bigg]; \quad (29c)$$

$$Q_r^1 = \frac{L}{\pi r}\left[\frac{1}{4} - \frac{r^2}{2ab} + \frac{\lambda_1(\rho^4 - u_1^4)}{2V_1^2} + \frac{\lambda_2(\rho^4 - u_2^4)}{2V_2^2} + \frac{d^2 r^2 \cos 2\theta}{2abf^2} - \sum_1^\infty n(2n+1) C_n r^{2n} \cos 2n\theta\right], \quad (30a)$$

$$Q_\theta^1 = \frac{L}{\pi r}\left[\rho^2\left(\frac{\lambda_1 u_1^2}{V_1^2} - \frac{\lambda_2 u_2^2}{V_2^2} - \frac{d^2}{2abt^2}\right) \sin 2\theta + \sum_1^\infty n(2n+1) C_n r^{2n} \sin 2n\theta\right]; \quad (30b)$$

$$\begin{matrix} M_r^2 \\ M_\theta^2 \end{matrix} = \frac{(1+\nu)L}{4\pi}\Bigg[1 \pm \tfrac{1}{2}\beta + \lambda_1 \ln V_1 + \lambda_2 \ln V_2$$

$$\mp 2\beta\left\{\frac{\lambda_1 u_1^2}{V_1^2}(\rho^2+u_1^2)\sin^2\theta + \frac{\lambda_2 u_2^2}{V_2^2}(\rho^2+u_2^2)\cos^2\theta\right\}$$

$$- C_0 - \sum_1^\infty\left\{(2n+1)(1 \pm n\beta)C_n \pm \frac{\beta n(2n-1)}{r^2}A_n\right\}r^{2n}\cos 2n\theta$$

$$- \ln\frac{|z+Z|}{a+b} + \frac{1\pm\beta}{f^2}\operatorname{Re}(zZ) \mp \frac{\beta d^2}{r^2 f^4}\operatorname{Re}(z^3 Z)\Bigg], \tag{31a, b}$$

$$M_{r\theta}^2 = \frac{(1-\nu)L}{4\pi}\Bigg[\left\{\frac{\lambda_1 u_1^2}{V_1^2}(u_1^2-\rho^2) - \frac{\lambda_2 u_2^2}{V_2^2}(u_2^2-\rho^2)\right\}\sin 2\theta$$

$$-\sum_1^\infty n\left\{(2n+1)C_n + \frac{2n-1}{r^2}A_n\right\}r^{2n}\sin 2n\theta + \frac{1}{f^2}\operatorname{Im}\left\{zZ\left(1-\frac{d^2 z^2}{f^2 r^2}\right)\right\}\Bigg]; \tag{31c}$$

$$Q_r^2 = \frac{L}{\pi r}\left[\frac{1}{4} + \frac{\lambda_1(\rho^4-u_1^4)}{2V_1^2} + \frac{\lambda_2(\rho^4-u_2^4)}{2V_2^2} - \sum_1^\infty n(2n+1)C_n r^{2n}\cos 2n\theta + \frac{1}{f^2}\operatorname{Re}(zZ)\right], \tag{32a}$$

$$Q_\theta^2 = \frac{L}{\pi r}\left[\left(\frac{\lambda_1 u_1^2}{V_1^2} - \frac{\lambda_2 u_2^2}{V_2^2}\right)\rho^2\sin 2\theta + \sum_1^\infty n(2n+1)C_n r^{2n}\sin 2n\theta - \frac{1}{f^2}\operatorname{Im}(zZ)\right], \tag{32}$$

where

$$\rho = r/c, \quad V_1^2 = \rho^4 + u_1^4 - 2u_1^2\rho^2\cos 2\theta, \quad V_2^2 = \rho^4 + u_2^4 + 2u_2^2\rho^2\cos 2\theta. \tag{33}$$

The real and imaginary parts of the quantities appearing in (13b), (30a, b, c) and (31a, b) can be easily computed by applying the following formulae:

$$|z+Z| = [T^2 + r^2 + \sqrt{2}r\sqrt{(T^2+r^2-f^2\cos 2\theta)}]^{1/2}, \tag{34}$$

$$\operatorname{Re}(z^n Z) = \frac{r^n}{\sqrt{2}}[\cos n\theta\sqrt{(T^2+r^2\cos 2\theta - f^2)} - \sin n\theta\sqrt{(T^2-r^2\cos 2\theta+f^2)}], \tag{35a}$$

$$\operatorname{Im}(z^n Z) = \frac{r^n}{\sqrt{2}}[\cos n\theta\sqrt{(T^2-r^2\cos 2\theta+f^2)} + \sin n\theta\sqrt{(T^2+r^2\cos 2\theta-f^2)}], \tag{35b}$$

where

$$n = 1, 3, \quad T^4 = r^4 + f^4 - 2f^2 r^2\cos 2\theta. \tag{36}$$

At the centre of the plate the shears vanish and

$$\begin{matrix} (M_r)_0 \\ (M_\theta)_0 \end{matrix} = \frac{(1+\nu)L}{4\pi}\Bigg[\tfrac{1}{2} - \ln\frac{t_1+t_2}{2} + \beta(\lambda_1 u_1^2 + \lambda_2 u_2^2 - \tfrac{1}{8}v^2) + 2\lambda_1 \ln u_1 + 2\lambda_2 \ln u_2$$

$$\pm \beta\cos 2\theta\left\{\lambda_1 - \lambda_2 - \tfrac{1}{2}m + \frac{1}{\kappa}(\lambda_2 u_2^4 - \lambda_1 u_1^4) + \frac{v^2 t^2}{16\kappa} - \frac{\kappa^2+3}{2\kappa}(\tfrac{1}{8}t^2 + \lambda_2 u_2^2 - \lambda_1 u_1^2)\right\}\Bigg], \tag{37a}$$

$$(M_{r\theta})_0 = \frac{(1-v)L\sin 2\theta}{4\pi}\left[\lambda_1 - \lambda_2 - \tfrac{1}{2}m + \frac{1}{\kappa}(\lambda_2 u_2^4 - \lambda_1 u_1^4) + \frac{v^2 t^2}{16\kappa} - \frac{\kappa^2+3}{2\kappa}\left(\tfrac{1}{8}t^2 + \lambda_2 u_2^2 - \lambda_1 u_1^2\right)\right]. \quad (37b)$$

5. NUMERICAL RESULTS

This section deals with the numerical and graphical representation of the deflections, moments and shears at various points in the first quadrant of the circular plate corresponding to specified dimensions of the uniformly loaded elliptic patch. We assume that

$$t_1 = a/c = 0{\cdot}7, \quad t_2 = b/c = 0{\cdot}5, \quad v = 0{\cdot}3.$$

Two different distributions of the point supports in the unloaded region are considered. In the first case we take $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$ while in the second case we take $u_1 = u_2 = 1$. In the latter case the point supports lie at the points of intersection of the major and minor axes with the edge of the plate. In each case we start by solving the system of the three linear simultaneous equations (1), (27a) and (27b), where the infinite series appearing in the last two equations are truncated after a sufficiently large number of terms. For the first case we find

$$\lambda_1 = 0{\cdot}1941, \quad \lambda_2 = 0{\cdot}3059, \quad A_0 = 0{\cdot}1926c^2. \quad (38a)$$

For the second case we find

$$\lambda_1 = 0{\cdot}2657, \quad \lambda_2 = 0{\cdot}2343, \quad A_0 = 1{\cdot}1434c^2. \quad (38b)$$

The deflection w, radial and transverse bending moments M_r, M_θ, twisting moment $M_{r\theta}$, shearing forces Q_r, Q_θ at any point (r, θ) may be put in the forms

$$w = \alpha pc^4/D, \quad M_r = \beta_1 pc^2, \quad M_\theta = \beta_2 pc^2, \quad M_{r\theta} = \beta_3 pc^2, \quad Q_r = \gamma_1 pc, \quad Q_\theta = \gamma_2 pc,$$

where α, β_1, β_2, β_3, γ_1 and γ_2 are dimensionless quantities. Introducing the values (38a, b) in (25a, b), (29a, b, c), (30a, b), (31a, b, c) and (32a, b) we obtain numerical values for the coefficients α, β_1, β_2, β_3, γ_1 and γ_2 which are listed in Tables 1–6. Graphs showing the variation of these coefficients along radii and circles concentric with the plate are plotted in Figs 2–25 for both cases.

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Table 1. Values of $10^4\alpha$ ($\nu = 0{\cdot}3$)

	$\theta = 0°$		$\theta = 30°$		$\theta = 60°$		$\theta = 90°$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	117	457	117	457	116	456	115	456
0·2	112	441	110	441	106	440	103	440
0·3	102	416	98	416	90	414	84	412
0·4	87	381	83	382	69	379	59	375
0·5	69	336	65	341	47	338	30	329
0·6	48	283	46	295	24	291	0	275
0·7	24	222	26	246	1	242	−26	214
0·8	0	154	7	197	−20	193	−50	148
0·9	−21	80	−11	148	−40	144	−73	77
1·0	−42	0	−28	101	−59	98	−97	0

Table 2. Values of $10^4\beta_1$ ($\nu = 0{\cdot}3$)

	$\theta = 0^\circ$		$\theta = 30^\circ$		$\theta = 60^\circ$		$\theta = 90^\circ$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	635	1358	691	1361	826	1389	905	1414
0·2	633	1327	642	1295	732	1312	822	1361
0·3	617	1274	567	1185	580	1184	672	1270
0·4	572	1194	472	1035	387	1008	436	1137
0·5	482	1085	356	845	192	786	50	958
0·6	332	942	221	619	47	541	$-\infty$	777
0·7	80	761	92	385	−7	337	−217	627
0.8	$-\infty$	580	23	202	−17	178	−73	494
0·9	−86	415	−2	75	−11	66	−17	361
1·0	0	0	0	0	0	0	0	0

Table 3. Values of $10^4\beta_2$ ($\nu = 0{\cdot}3$)

	$\theta = 0^\circ$		$\theta = 30^\circ$		$\theta = 60^\circ$		$\theta = 90^\circ$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	916	1414	852	1408	698	1374	609	1347
0·2	871	1360	833	1384	670	1348	535	1287
0·3	797	1271	799	1346	627	1303	405	1187
0.4	695	1145	749	1292	573	1240	202	1047
0.5	562	978	684	1225	502	1161	−129	863
0.6	383	765	607	1146	412	1072	$-\infty$	646
0·7	119	492	523	1063	333	998	−331	401
0·8	$-\infty$	139	454	997	275	939	−184	98
0·9	−115	−390	401	945	238	891	−131	−364
1·0	−77	$-\infty$	366	899	215	848	−115	$-\infty$

Table 4. Values of $10^4\beta_3$ ($\nu = 0{\cdot}3$)

	$\theta = 0^\circ$		$\theta = 30^\circ$		$\theta = 60^\circ$		$\theta = 90^\circ$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	0	0	−118	−21	−132	−33	0	0
0·2	0	0	−87	1	−138	−47	0	0
0·3	0	0	−42	34	−139	−69	0	0
0·4	0	0	6	78	−122	−97	0	0
0·5	0	0	44	127	−78	−125	0	0
0·6	0	0	66	174	−26	−151	0	0
0·7	0	0	70	209	−2	−174	0	0
0·8	0	0	60	219	−3	−185	0	0
0·9	0	0	50	211	−14	−183	0	0
1·0	0	0	45	191	−27	−170	0	0

Table 5. Values of $10^4\gamma_1$ ($\nu = 0{\cdot}3$)

	$\theta = 0°$		$\theta = 30°$		$\theta = 60°$		$\theta = 90°$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	−228	−427	−354	−461	−636	−536	−793	−577
0·2	−511	−867	−683	−916	−1235	−1067	−1654	−1167
0·3	−890	−1333	−980	−1358	−1732	−1584	−2690	−1782
0·4	−1394	−1840	−1256	−1779	−2020	−2082	−4181	−2437
0·5	−2071	−2411	−1515	−2169	−2041	−2552	−7469	−3152
0·6	−3078	−3086	−1749	−2515	−1391	−2376	$\mp\infty$	−3009
0·7	−5258	−3959	−1349	−2188	−718	−1904	3800	−3119
0·8	$\mp\infty$	−4419	−806	−1637	−352	−1449	1286	−3709
0·9	2041	−6908	−407	−1136	−160	−1023	494	−6005
1·0	213	$-\infty$	−137	−912	−47	−834	86	$-\infty$

Table 6. Values of $10^4\gamma_2$ ($\nu = 0{\cdot}3$)

	$\theta = 0°$		$\theta = 30°$		$\theta = 60°$		$\theta = 90°$	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
0·1	0	0	−236	−63	−253	−67	0	0
0·2	0	0	−419	−115	−559	−145	0	0
0·3	0	0	−506	−146	−956	−244	0	0
0·4	0	0	−477	−146	−1417	−374	0	0
0·5	0	0	−358	−111	−1757	−539	0	0
0·6	0	0	−215	−46	−1631	−599	0	0
0·7	0	0	49	209	−1252	−641	0	0
0·8	0	0	207	432	−911	−691	0	0
0·9	0	0	231	538	−682	−700	0	0
1.0	0	∞	191	437	−545	−538	0	$-\infty$

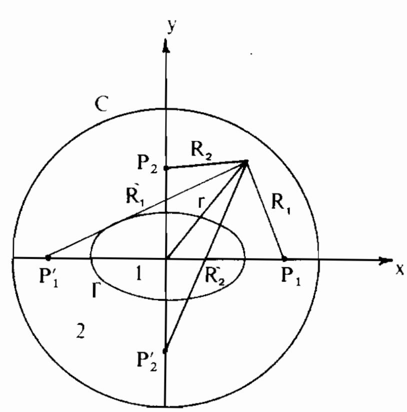

Fig. 1. The circular plate with the four supports at P_1, P_2, P'_1 and P'_2.

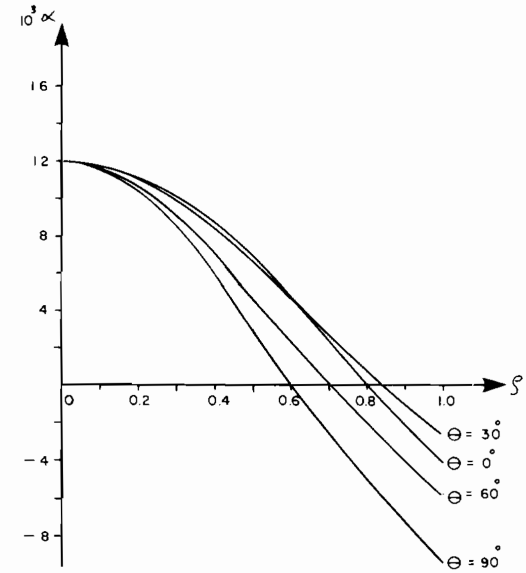

Fig. 2. Deflection profiles along radii of the circular plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

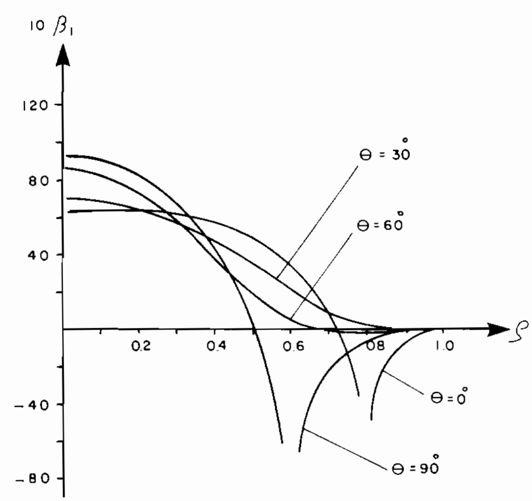

Fig. 3. Variation of radial bending moment factor along radii of the circular plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

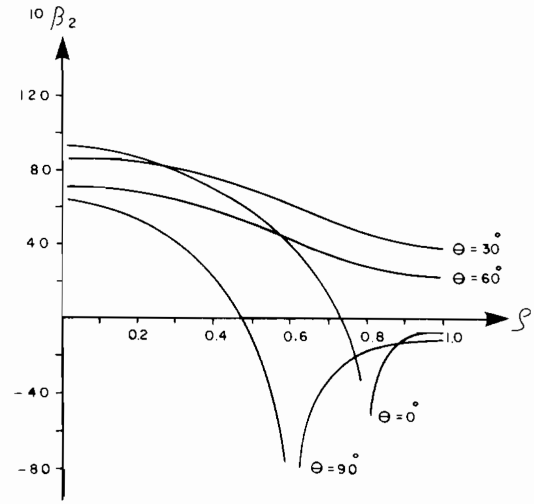

Fig. 4. Variation of transverse bending moment factor along radii of the circular plate, $u_1 = 0{\cdot}8, u_2 = 0{\cdot}6$.

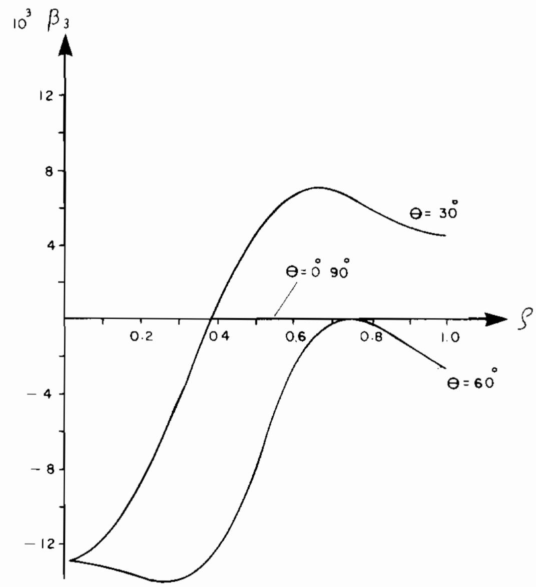

Fig. 5. Variation of twisting moment factor along radii of the circular plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

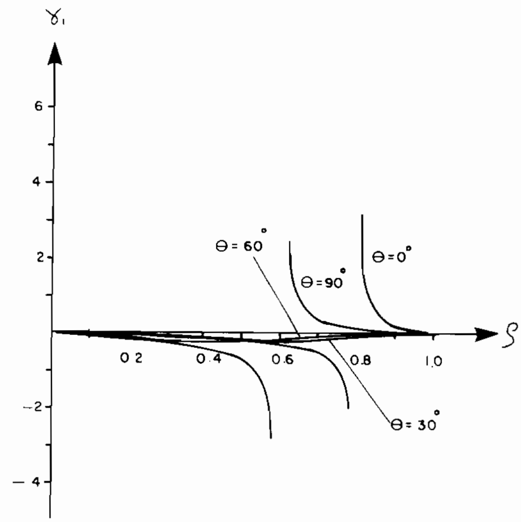

Fig. 6. Variation of radial shearing force factor along radii of the circular plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

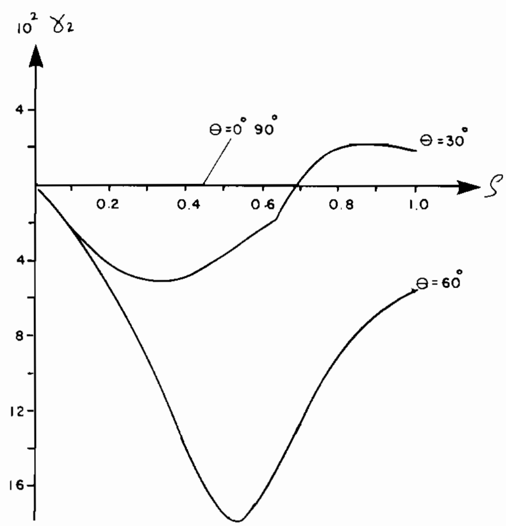

Fig. 7. Variation of transverse shearing force factor along radii of the circular plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

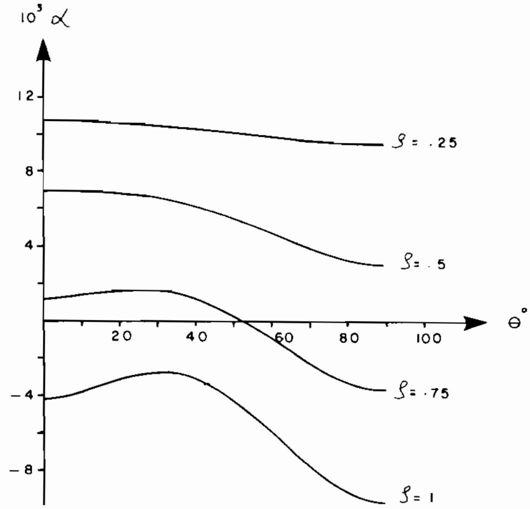

Fig. 8. Deflection profiles along semi-circles concentric with the plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

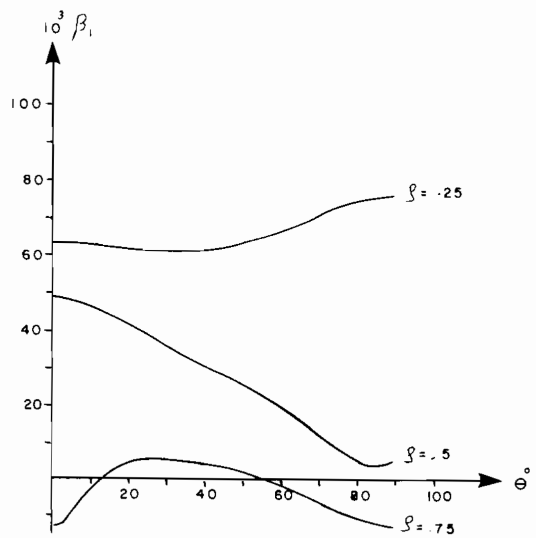

Fig. 9. Variation of radial bending moment factor along semi-circles of the plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

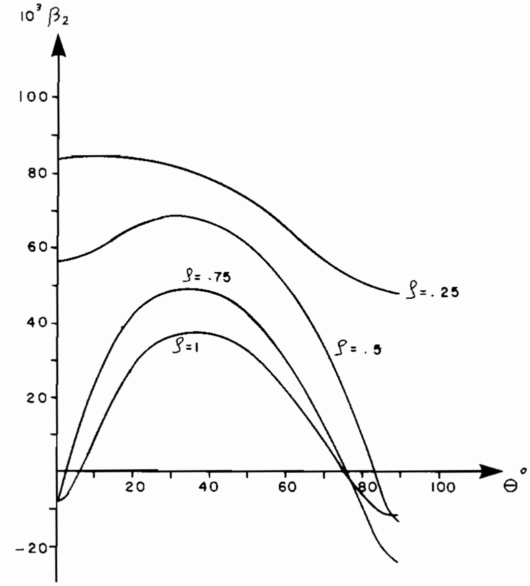

Fig. 10. Variation of transverse bending moment factor along semi-circles of the plate, $u_1 = 0{\cdot}8, u_2 = 0{\cdot}6$.

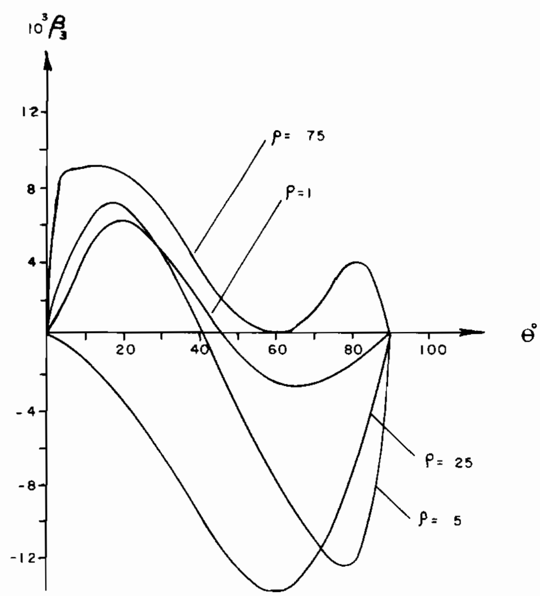

Fig. 11. Variation of twisting bending moment factor along sem-circles of the plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

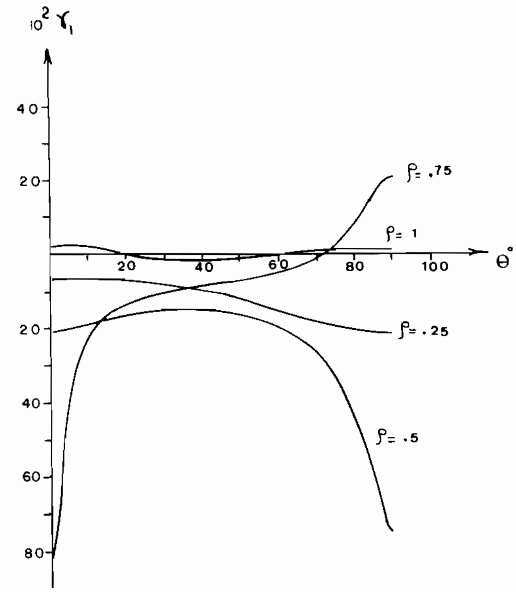

Fig. 12. Variation of radial shearing force factor along semi-circles of the plate, $u_1 = 0{\cdot}8$, $u_2 = 0{\cdot}6$.

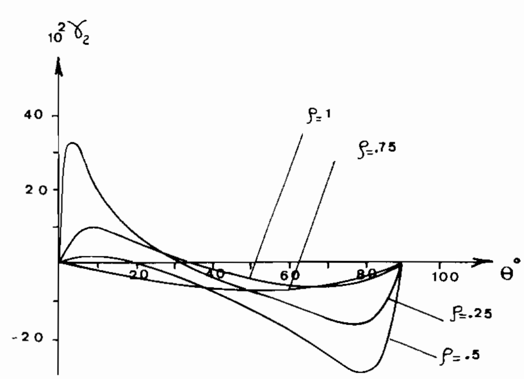

Fig. 13. Variation of transverse shearing force factor along semi-circles of the plate, $u_1 = 0{\cdot}8$, $u_2 = 0.6$.

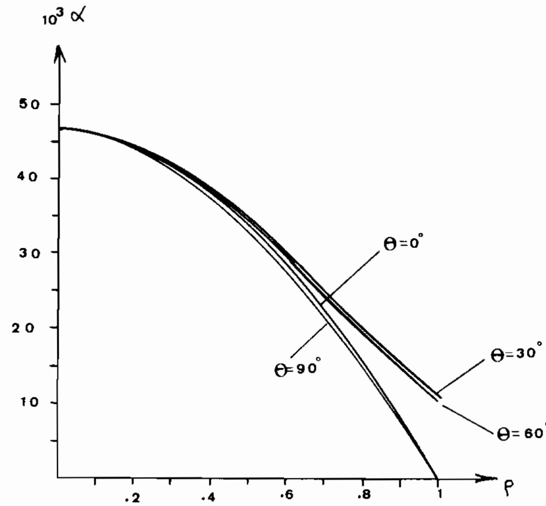

Fig. 14. Deflection profiles along radii of the circular plate, $u_1 = 1$, $u_1 = 1$.

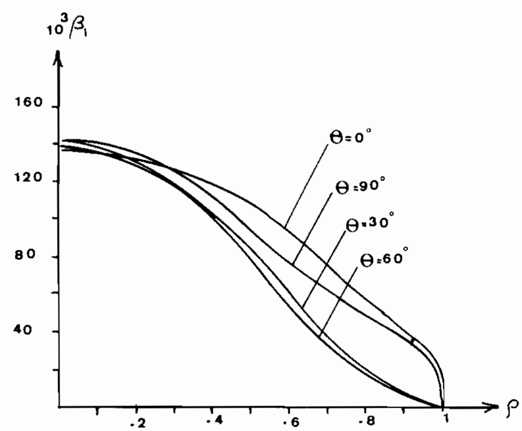

Fig. 15. Variation of radial bending moment factor along radii of the circular plate, $u_1 = 1$, $u_2 = 1$.

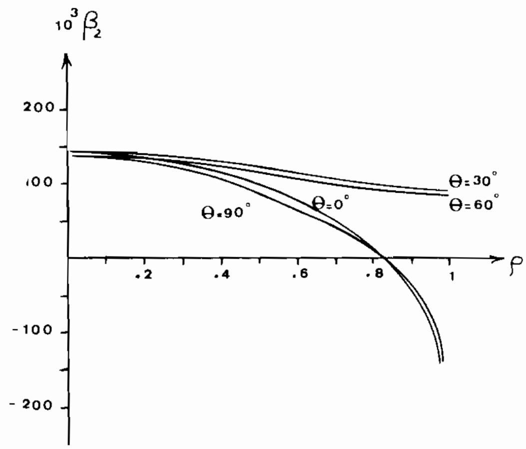

Fig. 16. Variation of transverse bending moment factor along radii of the circular plate, $u_1 = 1$, $u_2 = 1$.

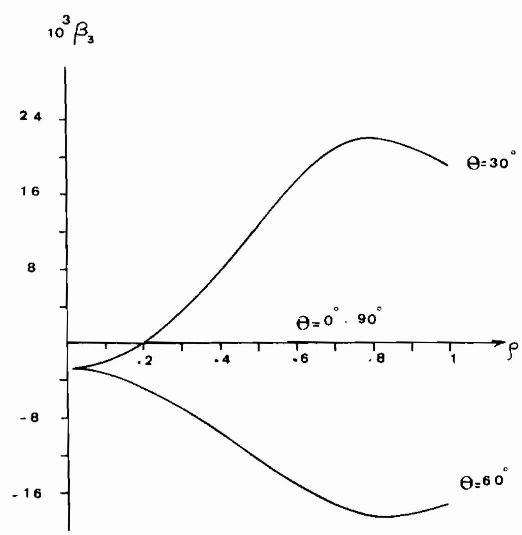

Fig. 17. Variation of twisting moment factor along radii of the circular plate, $u_1 = 1$, $u_2 = 1$.

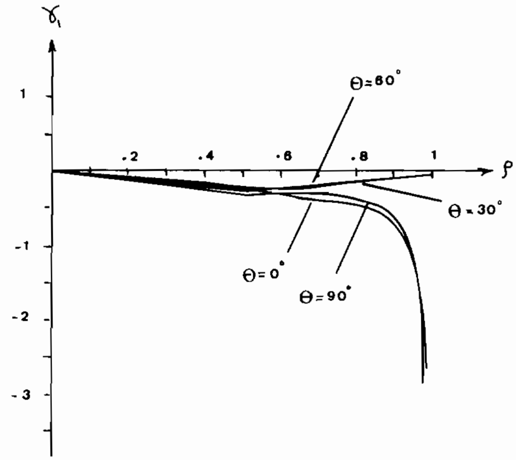

Fig. 18. Variation of radial shearing force factor along radii of the circular plate, $u_1 = 1$, $u_2 = 1$.

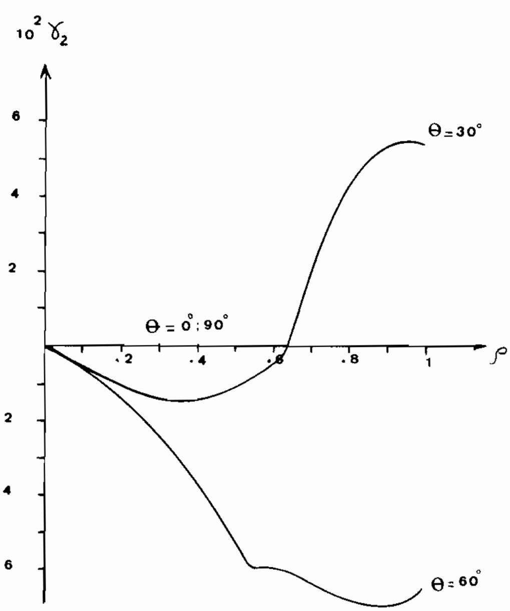

Fig. 19. Variation of transverse shearing force factor along radii of the circular plate, $u_1 = 1$, $u_2 = 1$.

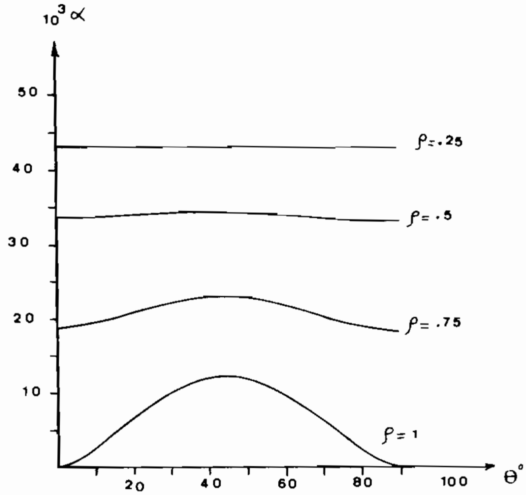

Fig. 20. Deflection profiles along semi-circles concentric with the plate, $u_1 = 1$, $u_2 = 1$.

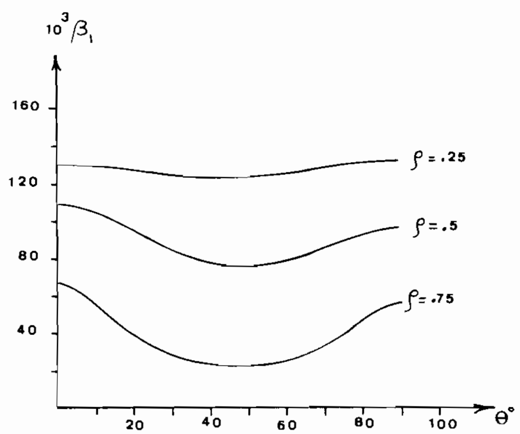

Fig. 21. Variation of radial bending moment factor along semi-circles of the plate, $u_1 = 1$, $u_2 = 1$.

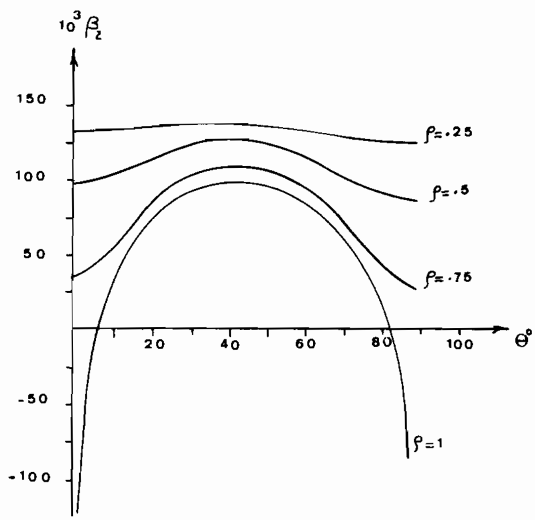

Fig. 22. Variation of transverse bending moment factor along semi-circles of the plate, $u_1 = 1$, $u_2 = 1$.

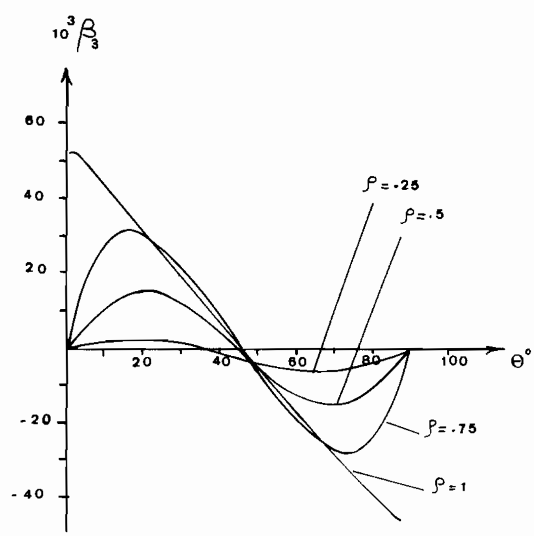

Fig. 23. Variation of twisting moment factor along semi-circles of the plate, $u_1 = 1$, $u_2 = 1$.

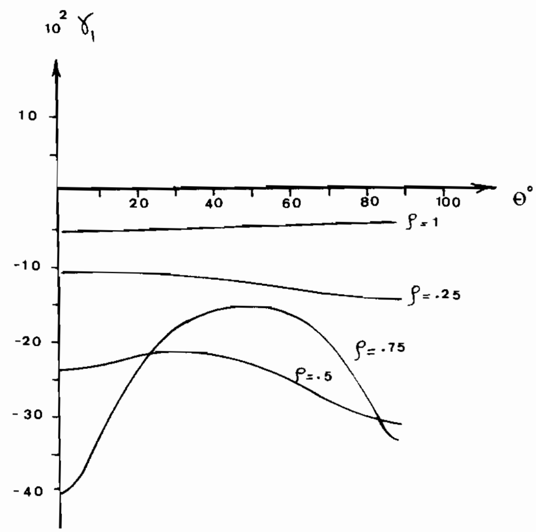

Fig. 24. Variation of radial shearing force factor along semi-circles of the plate, $u_1 = 1$, $u_2 = 1$.

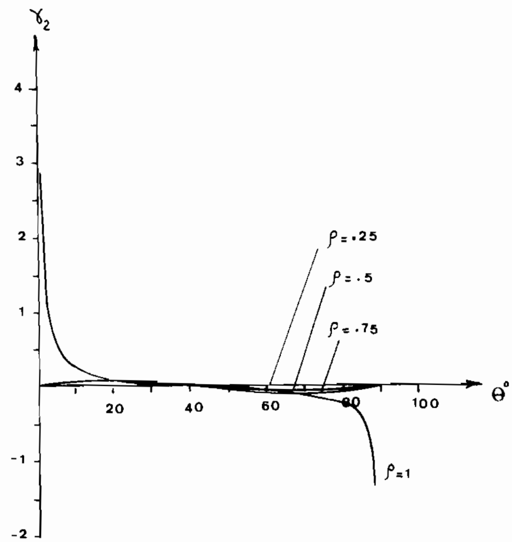

Fig. 25. Variation of transverse shearing force factor along semi-circles of the plate, $u_1 = 1$, $u_2 = 1$.

إنثناء صفيحة دائرية رقيقة محملة بإنتظام فوق قطع ناقص مركزي ومرتكزة بتماثل عند ٤ نقاط على محوريه

محمد نعيم يحيى أنور و وديع عطاالله بسالي
قسم الرياضيات بجامعة الكويت ، ص . ب ٥٩٦٩ ،
الصفاة ١٣٠٦٠ ، الكويت

خلاصة

حصل المؤلفان في هذا البحث على صيغ مضبوطة لمتسلسلات لانهائية للإزاحة العمودية الصغيرة التي تحدث عند أية نقطة من صفيحة دائرية رقيقة محملة بحمل منتظم موزع على مساحة قطع ناقص متحد مع الصفيحة في المركز ، عندما تكون الصفيحة مرتكزة إرتكازا متماثلا عند الرؤوس الأربعة لمعين ينطبق قطراه على محوري القطع . لقد أعطيت صيغ للعزوم والقوى القاصة واستخدم الحاسب الالكتروني في الحصول على جداول عددية لقيم الإزاحة والعزوم والقوى القاصة ، ومنحنيات لتمثيلها بيانيا .