J. Univ. Kuwait (Sci.) **14**, 1987

Cauchy–Riemann conditions for algebras isomorphic to the circulant algebra

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ABSTRACT

This paper generalizes the Cauchy–Riemann equations for complex differentiable functions to functions over certain n-dimensional commutative algebras over C with identity. Since the corresponding linear first order partial differential equations depend, *a priori*, on the basis chosen, this paper shows that in fact the set of functions which are differentiable is independent of the choice of basis. In the process, this paper generalizes to higher dimensions the notions of real and imaginary parts and of conjugation and the relationship between conjugation and multiplication for algebras over C^2.

1. INTRODUCTION

Closely related to the complex number field

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i^2 = -1\}$$

is the commutative algebra with identity

$$\mathbb{K} = \{x + ky \mid x, y \in \mathbb{R} \text{ and } k^2 = +1\}.$$

The algebra $\mathbb{K}$ plays an important role in the geometry of special relativity (see e.g. Yaglom 1979). These two algebras have natural matrix representations, i.e. circulant matrices for $\mathbb{K}$ and skew-circulant matrices for $\mathbb{C}$. The matrix representations induce natural multiplications on $\mathbb{R}^2$. As a result of these representations, a natural notion of differentiability for functions arises. It turns out that such a function is differentiable if its Jacobian matrix lies in the corresponding algebras. In this paper, we generalize these algebraic structures to higher dimensions and study the induced notions of differentiability. This will yield interesting product structures and differentiability structures on $\mathbb{R}^{2n}$. Even though these algebras on $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$) are distinct, we will show that they have the same set of differentiable functions. We shall also generalize to higher dimensions the relationship between the multiplications of $\mathbb{C}$ and $\mathbb{K}$ which were pointed out to us by L. Kauffman of Illinois–Chicago (private conversation).

2. INTRODUCTION TO ALGEBRAS

Let $\mathbb{F}$ denote either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. One can put two natural algebraic structures on $\mathbb{F}^2$ as follows: let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbb{F}$. For both algebras, define addition and conjugation as usual

$$z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \tag{1}$$

and

$$\overline{(x_i, y_i)} = (x_i, -y_i). \tag{2}$$

Define two multiplications on $\mathbb{F}^2$

$$z_1 \Delta_0 z_2 = (x_1 x_2 + y_1 y_2, x_1 y_2 + x_2 y_1) \tag{3}$$

and

$$z_1 \Delta_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \tag{4}$$

If $\mathbb{F} = \mathbb{R}$, then (4) is the usual multiplication for complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ where $i^2 = -1$. Equation (3) is the multiplication on $\mathbb{R}^2$ induced by $(x_1 + ky_1)(x_2 + ky_2)$ where $k^2 = +1$. We denote this second algebraic structure on $\mathbb{R}^2$ as $\mathbb{K}$. Yaglom (1979) uses this structure heavily in a geometry of special relativity. Leisenring (1979) makes intensive use of both algebraic structures on the bicomplex plane, where $\mathbb{F} = \mathbb{C}$. Indeed, $(\mathbb{R}^2, +, \Delta_0)$ and $(\mathbb{R}^2, +, \Delta_1)$ are not isomorphic, since the latter is a field but the former is not. However, Leisenring (1979) demonstrates that $(\mathbb{C}^2, +, \Delta_0)$ and $(\mathbb{C}^2, +, \Delta_1)$ are isomorphic. $(\mathbb{F}^2, +, \Delta_0)$ and $(\mathbb{F}^2, +, \Delta_1)$ are both isomorphic to certain matrix algebras. $(\mathbb{F}^2, +, \Delta_1)$ is algebraically isomorphic to

$$\left\{ \begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix} \middle| (x_1, y_1) \in \mathbb{F}^2 \right\},$$

the subspace of 2×2 skew-circulant matrices under matrix addition and multiplication. $(\mathbb{F}^2, +, \Delta_0)$ is algebraically isomorphic to

$$\left\{ \begin{pmatrix} x_1 & y_1 \\ y_1 & x_1 \end{pmatrix} \middle| (x_1, y_1) \in \mathbb{F}^2 \right\},$$

the subspace of 2×2 circulant matrices under matrix addition and multiplication. For more background on skew-circulant and circulant matrices, see Davis (1979) or Wilde (1983, 1986).

A special notion of differentiability for functions on $\mathbb{C}$ or $\mathbb{K}$ is induced by these matrix structures. A function f on $\mathbb{C}$ is $\mathbb{C}$-differentiable (i.e. holomorphic) if its Jacobian derivative on $\mathbb{R}^2$

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

is the corresponding algebra of skew-circulant matrices. This is Leisenring's approach to the Cauchy–Riemann conditions in $\mathbb{C}^2$: $u_x = v_y$ and $u_y = -v_x$. Furthermore, Leisenring (1979) proves that u and v are $\mathbb{C}$-differentiable if and only if

$$u(x, y) = \frac{1}{2}[f(x + iy) + g(x - iy)]$$

and

$$v(x, y) = \frac{1}{2i}[f(x + iy) - g(x - iy)],$$

where f and g are holomorphic functions on $\mathbb{C}$. Wilde (1983) demonstrates the same principle for $\mathbb{K}$-analytic functions: a function $\mathbb{K} \to \mathbb{K}$ is $\mathbb{K}$-differentiable if its Jacobian derivative as a map: $\mathbb{R}^2 \to \mathbb{R}^2$ is a circulant matrix. This time, the analogous Cauchy–Riemann conditions are: $u_x = v_y$ and $u_y = v_x$. Wilde (1983) proves that u and v are *differentiable* in the natural $(\mathbb{C}^2, +, \Delta_0)$ structure if and only if there exist $\mathbb{C}$-holomorphic functions f and g from $\mathbb{C}$ to $\mathbb{C}$ such that

$$u(x, y) = \frac{1}{2}[f(x+y) + g(x-y)]$$

$$v(x, y) = \frac{1}{2}[f(x+y) - g(x-y)].$$

We now construct the natural generalizations of these facts to higher dimensions. Let $\mathbf{k}_0, \ldots, \mathbf{k}_{n-1}$ be any basis of $\mathbb{C}^n$. For example, we could take the standard basis $\mathbf{e}_0, \ldots, \mathbf{e}_{n-1}$. Let ζ be the complex number $e^{2\pi i/n}$. Let

$$x = \sum_{h=0}^{n-1} x_h \mathbf{k}_h, \quad y = \sum_{h=0}^{n-1} y_h \mathbf{k}_h, \quad \text{and} \quad z = \sum_{h=0}^{n-1} z_h \mathbf{k}_h$$

be three vectors in $\mathbb{C}^n$, where x_h, y_h, z_h are all in $\mathbb{C}$. For each $l = 0, 1, \ldots, n-1$, we define an algebraic structure $(\mathbb{C}^n, \Delta_l)$ on $\mathbb{C}^n$, i.e. by

$$x + y = \sum_{h=0}^{n-1} (x_h + y_h)\mathbf{k}_h, \quad \text{and} \quad x\Delta_l y = \sum_{h=0}^{n-1}\sum_{j=0}^{n-1} \zeta^{lhj} x_h y_j \mathbf{k}_{h+j(\text{mod}\, n)}. \tag{5}$$

Note that $x\Delta_l \mathbf{k}_0 = x$ for all l.

If the base field is $\mathbb{R}$, for $n = 2$ and $l = 0$, (5) reduces to (3), and we have the multiplication for $\mathbb{K}$; for $n = 2$ and $l = 1$, (5) reduces to (4), and we have the multiplication for $\mathbb{C}$. Notice that for $l = 0$,

$$\left(\sum_{h=0}^{n-1} x_h \mathbf{k}_h\right)\Delta_0\left(\sum_{j=0}^{n-1} y_j \mathbf{k}_j\right) = \sum_{h=0}^{n-1}\sum_{j=0}^{n-1} x_h y_j k_{h+j(\text{mod}\, n)},$$

i.e. the subscripts on the $\mathbf{k}$'s behave like exponents modulo n. It follows that for any basis, $(\mathbb{C}^n, \Delta_0)$ is algebraically isomorphic to the algebra of $n \times n$ complex circulant matrices

$$B_n = \left\{ \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \cdots & x_{n-2} \\ \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_0 \end{pmatrix} \middle| (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{C}^n \right\}$$

with the usual matrix multiplication. B_n is a commutative algebra with identity over $\mathbb{C}$.

Proposition 1. For each $l = 0, 1, \ldots, n-1$, $(\mathbb{C}^n, \Delta_l)$ is a commutative algebra with identity over $\mathbb{C}$.

Proof. Multiplication is commutative by a simple change of indices. We show the associative property of Δ_l by verifying it for the basis vectors $\mathbf{k}_h$:

$$\begin{aligned} \mathbf{k}_h\Delta_l(\mathbf{k}_j\Delta_l\mathbf{k}_m) &= \mathbf{k}_h\Delta_l[\zeta^{ljm}\mathbf{k}_{j+m(\text{mod}\, n)}] \\ &= \zeta^{lh(j+m)}\zeta^{ljm}\mathbf{k}_{h+(j+m)(\text{mod}\, n)} \\ &= \zeta^{l(hj+hm+jm)}\mathbf{k}_{h+j+m(\text{mod}\, n)} \end{aligned}$$

is equal to

$$\begin{aligned}(\mathbf{k}_h\Delta_l\mathbf{k}_j)\Delta_l\mathbf{k}_m &= [\zeta^{lhj}\mathbf{k}_{h+j(\mathrm{mod}\,n)}]\Delta_l\mathbf{k}_m \\ &= \zeta^{lhj}\zeta^{l(h+j)m}\mathbf{k}_{(h+j)+m(\mathrm{mod}\,n)} \\ &= \zeta^{l(hj+hm+jm)}\mathbf{k}_{h+j+m(\mathrm{mod}\,n)}.\end{aligned}$$

Q.E.D.

To generalize the complex conjugation operation (2) on $\mathbb{F}^2$ to $(\mathbb{C}^n, \Delta_l)$, define

$$\theta(x) = \sum_{h=0}^{n-1} \zeta^h x_h \mathbf{k}_h. \tag{6}$$

θ is an automorphism on $(\mathbb{C}^n, \Delta_l)$, i.e. $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta x\Delta_l y) = \theta(x)\Delta_l\theta(y)$. Also,

$$\theta^j(x) = \sum_{h=0}^{n-1} \zeta^{hj} x_h \mathbf{k}_h \tag{7}$$

and $\theta^n(x) = x$.

Next we generalize $\mathrm{Re}(z)$ and $i\,\mathrm{Im}(z)$ to $(\mathbb{C}^n, \Delta_l)$. For $h = 0, 1, \ldots, n-1$, define the function $q_h : \mathbb{C}^n \to \mathbb{C}^n$ by

$$q_h = \frac{1}{n}\sum_{j=0}^{n-1} \zeta^{-hj}\theta^j. \tag{8}$$

Proposition 2. $q_0, q_1, \ldots, q_{n-1}$ have the following properties:

(i) $q_h(x) = x_h\mathbf{k}_h$;
(ii) $q_h^2 = q_h$;
(iii) $q_hq_j = \mathbf{0}$ for all $j \neq h$; and
(iv) $\sum_{h=0}^{n-1} q_h(x) = x$.

Proof.

(i)

$$\begin{aligned}q_h(x) &= \frac{1}{n}\sum_{j=0}^{n-1} \zeta^{-hj}\theta^j(x) \\ &= \frac{1}{n}\sum_{j=0}^{n-1} \zeta^{-hj}\left(\sum_{m=0}^{n-1} \zeta^{jm} x_m\mathbf{k}_m\right) \\ &= \frac{1}{n}\sum_{j=0}^{n-1}\sum_{m=0}^{n-1} \zeta^{j(-h+m)} x_m\mathbf{k}_m \\ &= \sum_{m=0}^{n-1}\left[\frac{1}{n}\sum_{j=0}^{n-1} \zeta^{j(-h+m)}\right] x_m\mathbf{k}_m \\ &= x_h\mathbf{k}_h\end{aligned}$$

because

$$\frac{1}{n}\sum_{j=0}^{n-1} \zeta^{j(-h+m)} = \begin{cases} 1 & \text{if} \quad m = h, \\ 0 & \text{if} \quad m \neq h. \end{cases}$$

(ii)

$$\begin{aligned}q_h(q_h(x)) &= q_h(x_h\mathbf{k}_h) \\ &= x_h\mathbf{k}_h \\ &= q_h(x).\end{aligned}$$

(iii) If $j \neq h$, $q_h(q_j(x)) = q_h(x_j \mathbf{k}_j) = \mathbf{0}$.

(iv)
$$\sum_{h=0}^{n-1} q_h(x) = \sum_{h=0}^{n-1} x_h \mathbf{k}_h = x.$$
Q.E.D.

Let us write $z_1 * z_2$ for $z_1 \Delta_1 z_2$, the multiplication in $\mathbb{C}$; and $z_1 z_2$ for $z_1 \Delta_0 z_2$, the multiplication in $\mathbb{K}$. Kauffman (private conversation) showed the symmetrical relationship between these two multiplications

(A)
$$z_1 * z_2 = \frac{1}{2}(z_1 z_2 + z_1 \overline{z_2} + \overline{z_1} z_2 - \overline{z_1 z_2})$$

and

(B)
$$z_1 z_2 = \frac{1}{2}(z_1 * z_2 + z_1 * \overline{z_2} + \overline{z_1} * z_2 - \overline{z_1} * \overline{z_2}).$$

Relation (A) can be rewritten in the form

$$\begin{aligned} z_1 \Delta_1 z_2 &= z_1 \frac{z_2 + \overline{z_2}}{2} + \overline{z_1} \frac{z_2 - \overline{z_2}}{2} \\ &= z_1 \operatorname{Re}(z_2) + \overline{z_1} k \operatorname{Im}(z_2) \\ &= \theta^0(z_1) \Delta_0 q_0(z_2) + \theta^1(z_1) \Delta_0 q_1(z_2). \end{aligned}$$

Similarly, (B) can be written as

$$z_1 \Delta_0 z_2 = \theta^0(z_1) \Delta_1 q_0(z_1) + \theta^1(z_1) \Delta_1 q_1(z_2).$$

For $n > 2$, we can generalize these formulas as follows.

Theorem 1.

$$\sum_{h=0}^{n-1} [\theta^h(x) \Delta_l q_h(\mathbf{y})] = \sum_{h=0}^{n-1} [q_h(x) \Delta_l \theta^h(\mathbf{y})] = x \Delta_{l+1} \mathbf{y},$$

for $l = 0, 1, \ldots, n-1$; and $l + 1$ taken modulo n.

Proof.

$$\begin{aligned} \sum_{h=0}^{n-1} [\theta^h(x) \Delta_l q_h(\mathbf{y})] &= \sum_{h=0}^{n-1} \left[\left(\sum_{j=0}^{n-1} \zeta^{hj} x_j \mathbf{k}_j \right) \Delta_l y_h \mathbf{k}_h \right] \\ &= \sum_{h=0}^{n-1} \sum_{j=0}^{n-1} \zeta^{ljh} \zeta^{hj} x_j y_h \mathbf{k}_{j+h(\text{mod}\, n)} \\ &= \sum_{j=0}^{n-1} \sum_{h=0}^{n-1} \zeta^{(l+1)jh} x_j y_h \mathbf{k}_{j+h(\text{mod}\, n)} \\ &= x \Delta_{l+1} \mathbf{y}. \end{aligned}$$

$\sum_{h=0}^{n-1} [q_h(x) \Delta_l \theta^h(\mathbf{y})] = x \Delta_{l+1} \mathbf{y}$ follows by symmetry because Δ_l and Δ_{l+1} are commutative. Q.E.D.

Therefore, the formulas in Theorem 1 are the generalizations of Kauffman's formulas [(A) and (B)] to $\mathbb{C}^n$ and give the relationships between the multiplications Δ_l.

3. IDEMPOTENT BASES

We will find it convenient to use an idempotent basis for these algebras. Each Δ_l has a different idempotent basis, expressible in terms of $\mathbf{k}_0, \ldots, \mathbf{k}_{n-1}$. To form these idempotent bases, note that by Equation (5),

$$\underbrace{\mathbf{k}_1 \Delta_l \cdots \Delta_l \mathbf{k}_1}_{j \text{ times}} = \zeta^{(1/2)lj(j-1)} \mathbf{k}_j \quad \text{for} \quad 1 \leqslant j \leqslant n-1. \tag{9}$$

and

$$\underbrace{\mathbf{k}_1 \Delta_l \cdots \Delta_l \mathbf{k}_1}_{n \text{ times}} = \zeta^{(1/2)ln(n-1)} \mathbf{k}_0, \tag{10}$$

where $\mathbf{k}_0$ is the identity for Δ_l $(0 \leqslant l \leqslant n-1)$. In addition, one can easily show

$$\zeta^{(1/2)ln(n-1)} = \begin{cases} 1 & \text{if } n \text{ is odd} \\ (-1)^l & \text{if } n \text{ is even.} \end{cases}$$

Now let

$$B = \begin{cases} e^{\pi i/n} & \text{if } n \text{ is even and } l \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases} \tag{11}$$

Then $B^n = \zeta^{(1/2)ln(n-1)}$,

$$\underbrace{\left(\frac{1}{B}\mathbf{k}_1\right)\Delta_l \cdots \Delta_l\left(\frac{1}{B}\mathbf{k}_1\right)}_{j \text{ times}} = B^{-j}\zeta^{(1/2)lj(j-1)}\mathbf{k}_j \tag{12}$$

and

$$\underbrace{\left(\frac{1}{B}\mathbf{k}_1\right)\Delta_l \cdots \Delta_l\left(\frac{1}{B}\mathbf{k}_1\right)}_{n \text{ times}} = \mathbf{k}_0. \tag{13}$$

Motivated by Equation (8), let

$$\begin{aligned} \mathbf{E}_{hl} &= \frac{1}{n}\sum_{j=0}^{n-1} \zeta^{-hj} \underbrace{\left(\frac{1}{B}\mathbf{k}_1\right)\Delta_l \cdots \Delta_l\left(\frac{1}{B}\mathbf{k}_1\right)}_{j \text{ times}} \\ &= \frac{1}{n}\sum_{j=0}^{n-1} B^{-j}\zeta^{(1/2)lj(j-1)-hj}\mathbf{k}_j. \end{aligned} \tag{14}$$

Then

$$\mathbf{E}_{hl}\Delta_l\mathbf{E}_{hl} = \mathbf{E}_{hl} \quad \text{for} \quad 0 \leqslant h \leqslant n-1; \tag{14.1}$$

$$\mathbf{E}_{hl}\Delta_l\mathbf{E}_{jl} = 0 \quad \text{if} \quad j \neq h; \tag{14.2}$$

and

$$\sum_{h=0}^{n-1} \mathbf{E}_{hl} = \mathbf{k}_0, \quad \text{the identity in } (\mathbb{C}^n, \Delta_l). \tag{14.3}$$

Thus $\mathbf{E}_{0l}, \ldots, \mathbf{E}_{n-1,l}$ is an idempotent basis for Δ_l. Solving for $\mathbf{k}_0, \ldots, \mathbf{k}_{n-1}$ in Equation (14) yields

$$\mathbf{k}_j = B^j\zeta^{-(1/2)lj(j-1)}\sum_{h=0}^{n-1}\zeta^{hj}\mathbf{E}_{hl} \tag{14.4}$$

for $0 \leqslant j \leqslant n-1$.

As an example, in $\mathbb{C}^2$,

$$B = \begin{cases} i & \text{if } l = 1 \\ 1 & \text{if } l = 0. \end{cases}$$

Equation (14) becomes $\mathbf{E}_{00} = \frac{1}{2}(\mathbf{k}_0 + \mathbf{k}_1)$, $\mathbf{E}_{10} = \frac{1}{2}(\mathbf{k}_0 - \mathbf{k}_1)$, $\mathbf{E}_{01} = \frac{1}{2}(\mathbf{k}_0 - i\mathbf{k}_1)$, and $\mathbf{E}_{11} = \frac{1}{2}(\mathbf{k}_0 + i\mathbf{k}_1)$. Equation (14.4) becomes $\mathbf{k}_0 = \mathbf{E}_{00} + \mathbf{E}_{10} = \mathbf{E}_{01} + \mathbf{E}_{11}$ and $\mathbf{k}_1 = \mathbf{E}_{00} - \mathbf{E}_{10} = i\mathbf{E}_{11}$ (see Leisenring 1979).

Now we want to derive elements $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}$ such that

$$\sum_{j=0}^{n-1} x_j \mathbf{k}_j = \sum_{h=0}^{n-1} \lambda_h \mathbf{E}_{hl}. \tag{15}$$

To do so, by (14.3) and (14.4),

$$\begin{aligned} \sum_{j=0}^{n-1} x_j \mathbf{k}_j &= \left(\sum_{h=0}^{n-1} \mathbf{E}_{hl} \right) \Delta_l \left(\sum_{j=0}^{n-1} x_j \mathbf{k}_j \right) \\ &= \sum_{h=0}^{n-1} \left[\sum_{j=0}^{n-1} B^j \zeta^{-(1/2)lj(j-1)+hj} x_j \right] \mathbf{E}_{hl}. \end{aligned}$$

Thus

$$\lambda_h = \sum_{j=0}^{n-1} B^j \zeta^{-(1/2)lj(j-1)+hj} x_j \quad (0 \leqslant h \leqslant n-1). \tag{16}$$

Solving for $x_0, x_1, \ldots, x_{n-1}$ results in

$$x_j = B^{-j} \zeta^{(1/2)lj(j-1)} \frac{1}{n} \sum_{h=0}^{n-1} \zeta^{-hj} \lambda_h \quad (0 \leqslant j \leqslant n-1). \tag{17}$$

In $\mathbb{C}^2$, Equations (16) are $\lambda_0 = x_0 + x_1$ and $\lambda_1 = x_0 - x_1$, if $l = 0$; $\lambda_0 = x_0 + ix_1$ and $\lambda_1 = x_0 - ix_1$, if $l = 1$. Also, Equations (17) are $x_0 = \frac{1}{2}(\lambda_0 + \lambda_1)$ and $x_1 = \frac{1}{2}(\lambda_0 - \lambda_1)$ if $l = 0$; $x_0 = \frac{1}{2}(\lambda_0 + \lambda_1)$ and $x_1 = (1/2i)(\lambda_0 - \lambda_1)$ if $l = 1$. λ_0 and λ_1 are the eigenvalues of $\begin{pmatrix} x_0 & x_1 \\ x_1 & x_0 \end{pmatrix}$ if $l = 0$. See Leisenring (1979) for an extensive discussion about the geometry of these facts in $\mathbb{C}^2$.

Using Equation (15), we show the following.

Theorem 2. $(\mathbb{C}^n, \Delta_l)$ is isomorphic to $(\mathbb{C}^n, \Delta_0)$ for $1 \leqslant l \leqslant n-1$.

Proof. We use the correspondence

$$\sum_{h=0}^{n-1} \lambda_h \mathbf{E}_{hl} \mapsto \sum_{h=0}^{n-1} \lambda_h \mathbf{E}_{h0}.$$

By Equations (14.1) and (14.2), if $\lambda_0^1, \lambda_1^1, \ldots, \lambda_{n-1}^1 \in \mathbb{C}$, then

$$\left(\sum_{h=0}^{n-1} \lambda_h \mathbf{E}_{hl} \right) \Delta_l \left(\sum_{h=0}^{n-1} \lambda_h^1 \mathbf{E}_{hl} \right) = \sum_{h=0}^{n-1} \lambda_h \lambda_h^1 \mathbf{E}_{hl}.$$

Also,

$$\left(\sum_{h=0}^{n-1} \lambda_h \mathbf{E}_{hl} \right) + \left(\sum_{h=0}^{n-1} \lambda_h^1 \mathbf{E}_{hl} \right) = \sum_{h=0}^{n-1} (\lambda_h + \lambda_h^1) \mathbf{E}_{hl}.$$

Since the last two equations also hold for $l = 0$, the correspondence above preserves $+$ and Δ_l. Q.E.D.

$(\mathbb{C}^n, \Delta_l)$ is isomorphic to $(\mathbb{C}^n, \Delta_0)$, and $(\mathbb{C}^n, \Delta_0)$ is isomorphic to B_n, the set of $n \times n$ complex circulant matrices. Thus $(\mathbb{C}^n, \Delta_l)$ is isomorphic to B_n.

Let $(\mathbb{C}^n, +, \cdot)$ be any algebraic structure on $\mathbb{C}^n$ which has an idempotent basis $\mathbf{E}_0, \mathbf{E}_1, \ldots, \mathbf{E}_{n-1}$ so that

$$\mathbf{E}_h^2 = \mathbf{E}_h \quad \text{for} \quad 0 \leqslant h \leqslant n-1; \tag{18.1}$$

$$\mathbf{E}_h \mathbf{E}_j = 0 \quad \text{if} \quad j \neq h; \tag{18.2}$$

and

$$\sum_{h=0}^{n-1} \mathbf{E}_h = I. \tag{18.3}$$

I being the identity for multiplication.

Now define n more bases $\mathbf{k}_{0l}, \mathbf{k}_{1l}, \ldots, \mathbf{k}_{n-1,l}$ $(0 \leqslant l \leqslant n-1)$ via (14.4), i.e.

$$\mathbf{k}_{jl} = B^j \zeta^{-(1/2)lj(j-1)} \sum_{h=0}^{n-1} \zeta^{hj} \mathbf{E}_h. \tag{19}$$

Then $\mathbf{k}_{0l} = \mathbf{I}$ (by (18.3) and (19)), and

$$\mathbf{k}_{Hl} \mathbf{k}_{jl} = \zeta^{lHj} \mathbf{k}_{H+j,l} \tag{20}$$

where $H + j$ is taken modulo n. Thus we have identified the idempotent bases and obtained n different bases $\mathbf{k}_{0,l}, \ldots, \mathbf{k}_{n-1,l}$ that satisfy Equation (5), identifying the multiplications $\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$ as one, namely . . This all follows from Theorem 2.

The fact that

$$\mathbf{k}_{Hl}\left(\sum_{j=0}^{n-1} x_j \mathbf{k}_{jl} \right) = \sum_{j=0}^{n-1} \zeta^{lHj} x_j \mathbf{k}_{H+j,l}$$

where $H + j$ is taken modulo n and $0 \leqslant H \leqslant n-1$, gives us a matrix representation of $(\mathbb{C}^n, +, \cdot)$, namely

$$[x, \mathbf{k}_{1l}x, \ldots, \mathbf{k}_{n-1,l}x]^T \tag{21}$$

represents $\sum_{j=0}^{n-1} x_j \mathbf{k}_{jl}$. If $l = 0$, the matrix representation is the algebra of circulant matrices. The quantity λ_h given by Equation (16) is an eigenvalue of (21) with

$$Y_h = (y_{0h}, y_{1h}, \ldots, y_{n-1,h})^T$$

where $y_{jh} = B^j \zeta^{-(1/2)lj(j-1)+hj}$ for $0 \leqslant j \leqslant n-1$, as the corresponding eigenvector.

4. DIFFERENTIABLE FUNCTIONS ON THE CIRCULANT ALGEBRA

In this section, we define multiplication on a basis of any commutative algebra over $\mathbb{C}$ and find its matrix representation, giving the algebra of circulant matrices as a special case. A function on the algebra is called differentiable if the transpose of its Jacobian is in the same form as the matrix representation of the algebra. The first-order, linear, partial differential equations the function satisfies are called CR conditions, since they generalize the classical Cauchy–Riemann conditions. We find the CR conditions for arbitrary bases of the circulant algebra and solve them, showing that each basis generates the same set of differentiable functions on B_n. Then we show that this differentiation on B_n satisfies the usual properties: linearity, the product rule, and chain rule.

Once we do this for all bases of B_n, we look at the special uses of the basis $\mathbf{k}_{0l}, \mathbf{k}_{1l}, \ldots, \mathbf{k}_{n-1,l}$, finding the CR conditions and solving them as special cases.

4.1. GENERALIZING THE CAUCHY–RIEMANN CONDITIONS

Let $\mathscr{A}_n$ denote an n-dimensional commutative algebra with identity I over $\mathbb{C}$ with a basis $\{a_0, a_1, \ldots, a_{n-1}\}$ where $a_0 = I$. Let the multiplication of $\mathscr{A}_n$ be defined by

$$a_s a_h = \sum_{m=0}^{n-1} c_{sh}^m a_m \quad \text{for} \quad 0 \leqslant s \leqslant n-1 \quad \text{and} \quad 0 \leqslant h \leqslant n-1 \tag{22}$$

where the m in c_{sh}^m is merely a superscript and $c_{sh}^m \in \mathbb{C}$ for all s, h, and m. Let the typical element of $\mathscr{A}_n$ be

$$X = \sum_{h=0}^{n-1} x_h a_h \tag{23}$$

where $x_0, x_1, \ldots, x_{n-1} \in \mathbb{C}$.

Then by Equation (22),

$$\begin{cases} a_s X = \displaystyle\sum_{h=0}^{n-1} x_h a_s a_h \\ \quad\;\; = \displaystyle\sum_{m=0}^{n-1} \left(\sum_{h=0}^{n-1} c_{sh}^m x_h \right) a_m. \end{cases} \tag{24}$$

A matrix representation of the algebra $\mathscr{A}_n$ is obtained by letting the row s be the coefficients of $a_0, a_1, \ldots, a_{n-1}$ in $a_s X$ given by Equation (24). If $x_h = 1$ and $x_j = 0$ for $j \neq h$, then the matrix represents a_h.

A special case of $\mathscr{A}_n$ is the algebra for which $a_s a_h = a_{s+h(\mathrm{mod}\, n)}$ where $a_0 = I$. If we let $a_1 = K$, then $a_h = K^h$, the hth power of K, and $K^n = I$. The matrix representation is an $n \times n$ complex circulant matrix

$$X = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{pmatrix}. \tag{25}$$

K^h is represented by the circulant matrix with $x_h = 1$ and $x_j = 0$ for $j \neq h$. From now on, let $\{I, K, K^2, \ldots, K^{n-1}\}$ be the basis of B_n. Then by Equation (25), $X = \sum_{h=0}^{n-1} x_h K^h$.

Let $u_0, u_1, \ldots, u_{n-1}$ be functions: $\mathbb{C}^n \to \mathbb{C}$. Now let

$$J^T = \begin{bmatrix} \dfrac{\partial u_0}{\partial x_0} & \dfrac{\partial u_1}{\partial x_0} & \dfrac{\partial u_2}{\partial x_0} & \cdots & \dfrac{\partial u_{n-1}}{\partial x_0} \\ \dfrac{\partial u_0}{\partial x_1} & \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \cdots & \dfrac{\partial u_{n-1}}{\partial x_1} \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{\partial u_0}{\partial x_{n-1}} & \dfrac{\partial u_1}{\partial x_{n-1}} & \dfrac{\partial u_2}{\partial x_{n-1}} & \cdots & \dfrac{\partial u_{n-1}}{\partial x_{n-1}} \end{bmatrix}. \tag{26}$$

This matrix is an element of the matrix representation of $\mathscr{A}_n$ if and only if

$$\frac{\partial u_m}{\partial x_s} = \sum_{h=0}^{n-1} c_{sh}^m \frac{\partial u_h}{\partial x_0} \quad \text{for} \quad 0 \leqslant m \leqslant n-1 \quad \text{and} \quad 1 \leqslant s \leqslant n-1. \tag{27}$$

Since Equations (27) are like the Cauchy–Riemann conditions, we call them the CR conditions. Indeed, they reduce to the Cauchy–Riemann conditions when $n = 2$, and $a_0 = 1$ and $a_1 = i$.

In the case of circulant matrices, the CR conditions are

$$\frac{\partial u_{h+s}}{\partial x_s} = \frac{\partial u_h}{\partial x_0} \quad \text{for} \quad 0 \leqslant s \leqslant n-1, \quad 0 \leqslant h \leqslant n-1, \quad \text{and where} \quad h+s \quad \text{is taken modulo } n. \tag{28}$$

4.2. AN IDEMPOTENT BASIS

The set B_n has an idempotent basis $\{E_0, E_1, \dots, E_{n-1}\}$ where if $\zeta = e^{2\pi i/n}$, then

$$E_j = \frac{1}{n} \sum_{h=0}^{n-1} \zeta^{-hj} K^h \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{29}$$

As discussed in Section 3, this basis has the properties that

$$E_j^2 = E_j \quad \text{for} \quad 0 \leqslant j \leqslant n-1; \tag{30a}$$

$$E_s E_j = 0 \quad \text{if} \quad s \neq j; \tag{30b}$$

$$\sum_{j=0}^{n-1} E_j = I; \tag{30c}$$

and

$$K^h = \sum_{j=0}^{n-1} \zeta^{hj} E_j \quad \text{for} \quad 0 \leqslant h \leqslant n-1. \tag{30d}$$

Let

$$\lambda_j = \sum_{h=0}^{n-1} \zeta^{hj} x_h \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{31}$$

Then

$$x_h = \frac{1}{n} \sum_{j=0}^{n-1} \zeta^{-hj} \lambda_j \quad \text{for} \quad 0 \leqslant h \leqslant n-1, \tag{32}$$

and

$$\sum_{h=0}^{n-1} x_h K^h = \sum_{j=0}^{n-1} \lambda_j E_j = X. \tag{33}$$

$\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ are the eigenvalues of the circulant matrix X, given by Equation (25).

Let $a_0, a_1, \dots, a_{n-1}$ where $a_0 = I$ be an arbitrary basis of B_n with the relation (22). Then there exists an invertible matrix (d_{jm}) with $d_{jm} \in \mathbb{C}$ such that

$$a_h = \sum_{j=0}^{n-1} d_{jh} E_j \quad \text{for} \quad 0 \leqslant h \leqslant n-1 \quad \text{and} \quad d_{j0} = 1 \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{34}$$

Then $a_s a_h = \sum_{j=0}^{n-1} d_{js} d_{jh} E_j$ for all s and h. But by Equation (22),

$$a_s a_h = \sum_{m=0}^{n-1} c_{sh}^m a_m = \sum_{m=0}^{n-1} c_{sh}^m \left(\sum_{j=0}^{n-1} d_{jm} E_j \right)$$
$$= \sum_{j=0}^{n-1} E_j \left(\sum_{m=0}^{n-1} c_{sh}^m d_{jm} \right).$$

So, since $E_0, E_1, \ldots, E_{n-1}$ are a basis,

$$d_{js} d_{jh} = \sum_{m=0}^{n-1} c_{sh}^m d_{jm} \quad \text{for} \quad 0 \leqslant j \leqslant n-1,$$
$$0 \leqslant s \leqslant n-1, \quad \text{and} \quad 0 \leqslant h \leqslant n-1. \tag{35}$$

Given the matrix (d_{jm}), we can use Equations (35) to solve for the c_{sh}^m's since (d_{jm}) is invertible.

4.3. SOLUTIONS OF THE CR CONDITIONS

Theorem 2 in Wilde (1983) stated that if $u_0, u_1, \ldots, u_{n-1}$ are entire functions: $\mathbb{C}^n \to \mathbb{C}$ that satisfy Equations (28), then there exist entire functions $f_0, f_1, \ldots, f_{n-1} : \mathbb{C} \to \mathbb{C}$ such that $u_0, u_1, \ldots, u_{n-1}$ are of the form

$$u_h = \frac{1}{n} \sum_{j=0}^{n-1} \zeta^{-hj} f_j \left(\sum_{l=0}^{n-1} \zeta^{jl} x_l \right) \quad \text{for} \quad 0 \leqslant h \leqslant n-1. \tag{36}$$

We can also prove that if $u_0, u_1, \ldots, u_{n-1}$ are given by Equations (36), then they satisfy Equations (28). We now generalize these results for all bases of B_n.

Theorem 3. Let the d_{jm}'s and the c_{sh}^m's be related by Equations (35). Let $u_0, u_1, \ldots, u_{n-1}$ be entire functions: $\mathbb{C}^n \to \mathbb{C}$. Then $u_0, u_1, \ldots, u_{n-1}$ satisfy the CR conditions if and only if there exist entire functions $f_0, f_1, \ldots, f_{n-1} : \mathbb{C} \to \mathbb{C}$ such that

$$\sum_{h=0}^{n-1} d_{jh} u_h = f_j \left(\sum_{l=0}^{n-1} d_{jl} x_l \right) \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{37}$$

Proof of sufficiency. Taking $\partial/\partial x_0$ of both sides of (37), we get

$$\sum_{h=0}^{n-1} d_{jh} \frac{\partial u_h}{\partial x_0} = f_j' \left(\sum_{l=0}^{n-1} d_{jl} x_l \right) \tag{38a}$$

because $d_{j0} = 1$ for $0 \leqslant j \leqslant n-1$. Multiplying both sides of (38a) by d_{js} gives us

$$\sum_{h=0}^{n-1} d_{js} d_{jh} \frac{\partial u_h}{\partial x_0} = d_{js} f_j' \left(\sum_{l=0}^{n-1} d_{jl} x_l \right). \tag{38b}$$

By Equation (35),

$$\sum_{h=0}^{n-1} d_{js} d_{jh} \frac{\partial u_h}{\partial x_0} = \sum_{h=0}^{n-1} \left(\sum_{m=0}^{n-1} c_{sh}^m d_{jm} \right) \frac{\partial u_h}{\partial x_0}$$
$$= \sum_{m=0}^{n-1} d_{jm} \left(\sum_{h=0}^{n-1} c_{sh}^m \frac{\partial u_h}{\partial x_0} \right). \tag{38c}$$

Taking $\partial/\partial x_s$ of both sides of (37) and changing the summation index from h to m results in

$$\sum_{m=0}^{n-1} d_{jm} \frac{\partial u_m}{\partial x_s} = d_{js} f_j'\left(\sum_{l=0}^{n-1} d_{jl} x_l\right). \tag{38d}$$

So by (38b), (38c), and (38d), we get

$$\sum_{m=0}^{n-1} d_{jm} \frac{\partial u_m}{\partial x_s} = \sum_{m=0}^{n-1} d_{jm}\left(\sum_{h=0}^{n-1} c_{sh}^m \frac{\partial u_h}{\partial x_0}\right) \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{38e}$$

Since (d_{jm}) is an invertible matrix, Equations (38e) imply the CR conditions.

Q.E.D.

Proof of necessity. Let

$$v_j = \sum_{m=0}^{n-1} d_{jm} u_m \quad \text{for} \quad 0 \leqslant j \leqslant n-1 \tag{39}$$

and

$$\lambda_t = \sum_{s=0}^{n-1} d_{ts} x_s \quad \text{for} \quad 0 \leqslant t \leqslant n-1. \tag{40}$$

We need to prove that v_j is a function of λ_j only.

Let D denote the inverse of the matrix (d_{jm}). Then Equations (40) imply that

$$x_s = \sum_{t=0}^{n-1} D_{st} \lambda_t \quad \text{for} \quad 0 \leqslant s \leqslant n-1. \tag{41a}$$

So by chain rule,

$$\frac{\partial u_m}{\partial \lambda_t} = \sum_{s=0}^{n-1} \frac{\partial x_s}{\partial \lambda_t} \frac{\partial u_m}{\partial x_s} = \sum_{s=0}^{n-1} D_{st} \frac{\partial u_m}{\partial x_s}, \tag{41b}$$

and by Equation (39),

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{m=0}^{n-1} d_{jm} \frac{\partial u_m}{\partial \lambda_t}. \tag{41c}$$

By Equation (41b),

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{m=0}^{n-1} d_{jm} \sum_{s=0}^{n-1} D_{st} \frac{\partial u_m}{\partial x_s}. \tag{41d}$$

Rearranging terms gives us

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{s=0}^{n-1} D_{st} \sum_{m=0}^{n-1} d_{jm} \frac{\partial u_m}{\partial x_s}. \tag{41e}$$

By the CR conditions,

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{s=0}^{n-1} D_{st} \sum_{m=0}^{n-1} d_{jm} \sum_{h=0}^{n-1} c_{sh}^m \frac{\partial u_h}{\partial x_0}. \tag{41f}$$

This is equivalent to

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{s=0}^{n-1} D_{st} \sum_{h=0}^{n-1} \left(\sum_{m=0}^{n-1} c_{sh}^m d_{jm}\right) \frac{\partial u_h}{\partial x_0}. \tag{41g}$$

Applying Equation (35) gives us

$$\frac{\partial v_j}{\partial \lambda_t} = \sum_{s=0}^{n-1} D_{st} \sum_{h=0}^{n-1} d_{js} d_{jh} \frac{\partial u_h}{\partial x_0}. \tag{41h}$$

If we rearrange terms, we have

$$\frac{\partial v_j}{\partial \lambda_t} = \left(\sum_{s=0}^{n-1} d_{js} D_{st}\right)\left(\sum_{h=0}^{n-1} d_{jh} \frac{\partial u_h}{\partial x_0}\right). \tag{41i}$$

Since (d_{jm}) and (D_{st}) are inverse matrices,

$$\sum_{s=0}^{n-1} d_{js} D_{st} = \begin{cases} 1 & \text{if} \quad t = j \\ 0 & \text{if} \quad t \neq j. \end{cases} \tag{42}$$

Also, by Equation (39),

$$\frac{\partial v_j}{\partial x_0} = \sum_{h=0}^{n-1} d_{jh} \frac{\partial u_h}{\partial x_0}. \tag{42a}$$

Therefore,

$$\frac{\partial v_j}{\partial \lambda_t} = \begin{cases} \dfrac{\partial v_j}{\partial x_0} & \text{if} \quad t = j \\ 0 & \text{if} \quad t \neq j. \end{cases} \tag{43}$$

Now Equation (43) says that $\partial v_j / \partial \lambda_j = \partial v_j / \partial x_0$. In deriving this equation, λ_j depended on x_0. Thus for each j, v_j is an entire function of λ_j only out of all the λ_t's. Q.E.D.

Thus Equations (37) give all entire functions $u_0, u_1, \ldots, u_{n-1} : \mathbb{C}^n \to \mathbb{C}$ that satisfy the CR conditions (Equations 27). Are the differentiable functions on B_n the same no matter which basis is used?

Theorem 4. No matter which basis of B_n we use, the CR conditions of that basis lead to the same set of differentiable functions of B_n.

Proof. By Equation (34),

$$\begin{aligned} \sum_{h=0}^{n-1} x_h a_h &= \sum_{h=0}^{n-1} x_h \left(\sum_{j=0}^{n-1} d_{jh} E_j\right) \\ &= \sum_{j=0}^{n-1} \left(\sum_{h=0}^{n-1} d_{jh} x_h\right) E_j, \end{aligned} \tag{44}$$

Similarly,

$$\sum_{h=0}^{n-1} u_h a_h = \sum_{j=0}^{n-1} \left(\sum_{h=0}^{n-1} d_{jh} u_h\right) E_j.$$

Thus by (37),

$$\sum_{h=0}^{n-1} u_h a_h = \sum_{j=0}^{n-1} f_j\left(\sum_{l=0}^{n-1} d_{jl} x_l\right) E_j. \tag{45}$$

Finally, by Equation (40) with a change of indices,

$$\sum_{h=0}^{n-1} u_h a_h = \sum_{j=0}^{n-1} f_j(\lambda_j) E_j \tag{46}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $\sum_{h=0}^{n-1} x_h K^h$. Thus, no matter which basis $a_0, a_1, \ldots, a_{n-1} \in B_n$ we use, we obtain as differentiable functions those of the form $\sum_{j=0}^{n-1} f_j(\lambda_j) E_j$ where $f_0, f_1, \ldots, f_{n-1}$ are entire functions: $\mathbb{C} \to \mathbb{C}$. Q.E.D.

Also, the transpose of the Jacobian of $u_0, u_1, \ldots, u_{n-1}$ is the matrix representation of

$$\sum_{h=0}^{n-1} \frac{\partial u_h}{\partial x_0} a_h = \sum_{j=0}^{n-1} \left(\sum_{h=0}^{n-1} d_{jh} \frac{\partial u_h}{\partial x_0} \right) E_j$$
$$= \sum_{j=0}^{n-1} f_j' \left(\sum_{l=0}^{n-1} d_{jl} x_l \right) E_j \tag{47}$$
$$= \sum_{j=0}^{n-1} f_j'(\lambda_j) E_j$$

by Equations (44), (38a), and (40). Thus we can call all sides of (47) the derivative of (45) and (46). It is obtained by differentiating the f_j's.

Note that this differentiation is linear and obeys the product rule and chain rule. To prove the latter statement, let

$$u = \sum_{j=0}^{n-1} f_j(\lambda_j) E_j \quad \text{and} \quad v = \sum_{j=0}^{n-1} g_j(\lambda_j) E_j$$

where λ_j is given by Equation (40), and $f_0, f_1, \ldots, f_{n-1}, g_0, g_1, \ldots, g_{n-1}$ are entire functions: $\mathbb{C} \to \mathbb{C}$. Let

$$c = \sum_{j=0}^{n-1} c_j E_j$$

where $c_0, c_1, \ldots, c_{n-1}$ are constants in $\mathbb{C}$. Then

$$u' = \sum_{j=0}^{n-1} f_j'(\lambda_j) E_j, \quad v' = \sum_{j=0}^{n-1} g_j'(\lambda_j) E_j, \quad \text{and} \quad c' = 0.$$

Also,

$$u + v = \sum_{j=0}^{n-1} (f_j + g_j) E_j;$$

$$cu = \sum_{j=0}^{n-1} (c_j f_j)(\lambda_j) E_j;$$

$$uv = \sum_{j=0}^{n-1} (f_j g_j)(\lambda_j) E_j;$$

and

$$u \cdot v = \sum_{j=0}^{n-1} (f_j \cdot g_j)(\lambda_j) E_j.$$

From these, we can show that

$$(u + v)' = u' + v', \quad (cu)' = cu', \quad (uv)' = uv' + u'v,$$

and

$$(u \cdot v)' = (u' \cdot v)v'.$$

4.4. SPECIAL CASES

Taking the matrix representation (21) of $\sum_{j=0}^{n-1} x_j \mathbf{k}_{jl}$ (in Section 3), if the transpose of the Jacobian matrix of functions $u_0, u_1, \ldots, u_{n-1} : \mathbb{C}^n \to \mathbb{C}$ is of the form of this matrix representation, then the CR conditions take the form

$$\frac{u_{h+s}}{\partial x_s} = \zeta^{lhs}\frac{\partial u_h}{\partial x_0} \quad \text{for} \quad 0 \leqslant h \leqslant n-1, \quad 0 \leqslant s \leqslant n-1, \tag{48}$$

where $h+s$ is taken modulo n. If

$$u\left(\sum_{h=0}^{n-1} x_h \mathbf{k}_{hl}\right) = \sum_{h=0}^{n-1} u_h \mathbf{k}_{hl}, \tag{49}$$

then

$$u' = \lim_{\Delta x_s \to 0} \frac{u[x_0\mathbf{k}_{0l} + \cdots + (x_s + \Delta x_s)\mathbf{k}_{sl} + \cdots + x_{n-1}\mathbf{k}_{n-1,l}] - u}{\Delta x_s \mathbf{k}_{sl}}. \tag{50}$$

This is well-defined because the multiplicative inverse of $\mathbf{k}_{sl}$ is $\zeta^{ls^2}\mathbf{k}_{n-s,l}$. Thus (50) becomes

$$u' = \zeta^{ls^2}\mathbf{k}_{n-s,l}\sum_{h=0}^{n-1}\frac{\partial u_h}{\partial x_s}\mathbf{k}_{hl}$$

$$= \sum_{h=0}^{n-1}\zeta^{-l(h-s)s}\frac{\partial u_h}{\partial x_s}\mathbf{k}_{h-s,l}$$

where $h-s$ is taken modulo n. Replacing h by $h+s$ (modulo n), we obtain

$$u' = \sum_{h=0}^{n-1}\zeta^{-lhs}\frac{\partial u_{h+s}}{\partial x_s}\mathbf{k}_{hl}.$$

By (48), all these derivatives of u in the directions of the n axes are equal. Thus

$$u' = \sum_{h=0}^{n-1}\frac{\partial u_h}{\partial x_0}\mathbf{k}_{hl} = \sum_{h=0}^{n-1}\zeta^{-lhs}\frac{\partial u_{h+s}}{\partial x_s}\mathbf{k}_{hl}. \tag{51}$$

As a special case of Theorem 3, $u_0, u_1, \ldots, u_{n-1}$ are entire functions: $\mathbb{C}^n \to \mathbb{C}$ satisfying Equations (48) if and only if $f_0, f_1, \ldots, f_{n-1}$ are entire functions: $\mathbb{C} \to \mathbb{C}$ such that

$$u_j = B^{-j}\zeta^{(1/2)lj(j-1)}\frac{1}{n}\sum_{h=0}^{n-1}\zeta^{-hj}f_h\left(\sum_{m=0}^{n-1}B^m\zeta^{mh-(1/2)lm(m-1)}x_m\right) \tag{52}$$

for $0 \leqslant j \leqslant n-1$. Also,

$$\sum_{j=0}^{n-1}u_j\mathbf{k}_{jl} = \sum_{h=0}^{n-1}f_h\left(\sum_{m=0}^{n-1}B^m\zeta^{mh-(1/2)lm(m-1)}x_m\right)\mathbf{E}_h$$

$$= \sum_{h=0}^{n-1}f_h(\lambda_h)\mathbf{E}_h \tag{53}$$

where λ_h is given by Equations (16).

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(Received 8 September 1985, revised 30 September 1986)

شروط كوشي ـ ريمان للجبريات المماثلة للجبر الدائري

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الولايات المتحدة الأمريكية

خلاصة

في هذا البحث تعميم لمعادلات كوشي ـ ريمان لدوال معقدة قابلة التفاضل على بعض الجبريات الإبدالية في الحقل المركب C ذي الوحدة . وبما أن المعادلات التفاضلية الخطية من الرتبة الأولى المناظرة تعتمد ، سلفا ، على القاعدة المنتخبة ، فان هذا البحث يظهر في الحقيقة أن مجموعة الدوال قابلة التفاضل مستقلة عن القاعدة المنتخبة . وأثناء البحث ، أمكن تعميم مفهومي الجزئين الحقيقي والتخيلي ، وكذلك مفهومة الترافق والعلاقة بين الترافق والضرب للجبريات على C^2 .