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Fixed point theorems for multifunctions satisfying a rational inequality

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ABSTRACT

We prove two fixed point theorems for a sequence of multifunctions satisfying a rational inequality which generalize Theorem 1 of Fisher (1980).

INTRODUCTION

Let (X, d) be a metric space. We denote by $\mathrm{CB}(X)$ the set of all nonempty closed bounded subsets of (X, d) and by H the Hausdorff–Pompeiu metric on $\mathrm{CB}(X)$

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B); \sup_{y \in B} d(y, A)\right)$$

where $A, B \in \mathrm{CB}(X)$ and

$$d(x, A) = \inf_{y \in A} \{d(x, y)\}.$$

Let $A, B \in \mathrm{CB}(X)$ and $k > 1$. In what follows, the following well-known fact will be used: For each $a \in A$, there is a $b \in B$ such that

$$d(a, b) \leqslant k \cdot H(A, B).$$

Let (X, d) be a metric space, we denote

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$$

where $A, B \in \mathrm{CB}(X)$. If A consists of a single point 'a' we write $\delta(A, B) = \delta(a, B)$. If $\delta(A, B) = 0$ then $A = B = \{a\}$ (Lemma 1, Rus 1975). Let $T : X \to X$ be a multifunction. Denote

$$F(T) = \{x \in X : x \in Tx\}.$$

In Fisher (1980), mappings S and T of a complete metric space (X, d) into itself were considered satisfying the inequality

$$d(Sx, Ty) \leqslant \frac{b \cdot d(x, Sx)d(x, Ty) + c \cdot d(y, Ty)d(y, Sx)}{d(x, Sx) + d(y, Ty)} \tag{1}$$

when $d(x, Sx) + d(y, Ty) \neq 0$ and the equality

$$d(Sx, Ty) = 0 \tag{2}$$

when $d(x, Sx) + d(y, Ty) = 0$ for all x, y in X, where $b, c \geqslant 0$ and $bc < 1$.

The object of this paper is to consider the replacement of S and T by multifunctions S and T into $\mathrm{CB}(X)$ and prove a fixed point theorem for two malfunctions T_1 and T_2 as well as for a sequence of multifunctions which generalizes Theorem 1 of Fisher (1980).

The method used is a combination of methods used in Achari (1977), Fisher (1980) and Kita (1977).

MAIN THEOREM

Lemma 1. Let (X, d) be a metric space and $T_1, T_2 : (X, d) \to \mathrm{CB}(X)$ be two multifunctions with $F(T_1) \neq \varnothing$. If $p \geqslant 1$, $s \geqslant 1$, $b \geqslant 0$, $c \geqslant 0$, $bc < 1$ and if

$$H^p(T_1x, T_2y) \leqslant \frac{b \cdot d^p(x, T_1x)d^s(x, T_2y) + c \cdot d^p(y, T_2y)d^s(y, T_1x)}{[\delta(x, T_1x) + \delta(y, T_2y)]^s} \tag{3}$$

holds for all $x, y \in X$ for which $\delta(x, T_1x) + \delta(y, T_2y) \neq 0$, then $F(T_2) \neq \varnothing$ and $F(T_1) = F(T_2)$.

Proof. Let $u \in F(T_1)$, then $u \in T_1u$ and if $d(u, T_2u) \neq 0$ then by (3) we have

$$\begin{aligned} d^p(u, T_2u) &\leqslant H^p(T_1u, T_2u) \\ &\leqslant \frac{b \cdot d^p(u, T_1u)d^s(u, T_2u) + c \cdot d^p(u, T_2u)d^s(u, T_1u)}{[\delta(u, T_1u) + \delta(u, T_2u)]^s} \\ &\leqslant \frac{b \cdot d^p(u, T_1u)d^s(u, T_2u) + c \cdot d^p(u, T_2u)d^s(u, T_1u)}{[d(u, T_1u) + d(u, T_2u)]^s} \end{aligned}$$

which implies $d(u, T_2u) = 0$. Since T_2u is closed this shows that $u \in T_2u$, which implies $F(T_1) \subset F(T_2)$. Analogously, $F(T_2) \subset F(T_1)$.

Theorem 1. Let (X, d) be a complete metric space and $T_1, T_2 : X \to \mathrm{CB}(X)$ two multifunctions such that for all $x, y \in X$ the inequality (3) holds if $\delta(x, T_1x) + \delta(y, T_2y) \neq 0$ where $p \geqslant 1, s \geqslant 1, b \geqslant 0, c \geqslant 0$ and $bc < 1$. Then T_1 and T_2 have common fixed points and $F(T_1) = F(T_2)$.

Proof. Choose a real number k with

$$1 < k < \left(\frac{1}{bc}\right)^{1/2p}. \tag{4}$$

Let $x_0 \in X$ and $x_1 \in T_1x_0$. Then there is an $x_2 \in T_2x_1$ such that $d(x_1, x_2) \leqslant kH(T_1x_0, T_2x_1)$. Suppose $x_3, x_4, \ldots, x_{2n-1}, x_{2n}, \ldots$ could be chosen so that $x_{2n-1} \in T_1x_{2n-2}$, $x_{2n} \in T_2x_{2n-1}$ and

$$d(x_{2n-1}, x_{2n}) \leqslant kH(T_1x_{2n-2}, T_2x_{2n-1})$$

$$d(x_{2n-2}, x_{2n-1}) \leqslant kH(T_1x_{2n-2}, T_2x_{2n-3}).$$

Suppose first of all that

$$\delta(x_{2n-2}, T_1x_{2n-2}) + \delta(x_{2n-1}, T_2x_{2n-1}) = 0$$

for some n. Then $x_{2n-2} = \{T_1x_{2n-2}\} = x_{2n-1} = \{T_2x_{2n-1}\}$ and $x_{2n-2} = x_{2n-1}$ is a common fixed point for T_1 and T_2. Similarly

$$\delta(x_{2n-1}, T_2x_{2n-1}) + \delta(x_{2n}, T_1x_{2n}) = 0$$

for some n implies that $x_{2n-1} = x_{2n}$ is a common fixed point for T_1 and T_2. Now suppose that

$$\delta(x_{2n-2}, T_1x_{2n-2}) + \delta(x_{2n-1}, T_2x_{2n-1}) \neq 0$$

for $n = 1, 2, \ldots$. Then by (3) we have successively

$$\begin{aligned}
d^p(x_{2n-1}, x_{2n}) &\leqslant k^pH^p(T_1x_{2n-2}, T_2x_{2n-1})\\
&\leqslant k^p \cdot \frac{b \cdot d^p(x_{2n-2}, T_1x_{2n-2})d^s(x_{2n-2}, T_2x_{2n-1}) + c \cdot d^p(x_{2n-1}, T_2x_{2n-1}) \cdot d^s(x_{2n-1}, T_1x_{2n-2})}{[\delta(x_{2n-2}, T_1x_{2n-2}) + \delta(x_{2n-1}, T_2x_{2n-1})]^s}\\
&\leqslant \frac{bk^pd^p(x_{2n-2}, x_{2n-1})d^s(x_{2n-2}, x_{2n})}{[d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})]^s}\\
&\leqslant \frac{bk^pd^p(x_{2n-2}, x_{2n-1})[d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})]^s}{[d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})]^s}\\
&= k^pbd^p(x_{2n-2}, x_{2n-1}).
\end{aligned}$$

Thus $d(x_{2n-1}, x_{2n}) \leqslant k \cdot b^{1/p} \cdot d(x_{2n-2}, x_{2n-1})$ holds.

Similarly we have

$$d(x_{2n}, x_{2n+1}) \leqslant k \cdot c^{1/p} \cdot d(x_{2n-1}, x_{2n})$$

holds too.

Repeating the above argument, we obtain

$$d(x_{2n+1}, x_{2n}) \leqslant [k^2(bc)^{1/p}]^n \cdot d(x_1, x_0).$$

Since $0 < k^2(bc)^{1/p} < 1$, by (4), by a routine calculation one can show that $\{x_n\}$ is a Cauchy sequence and since X is complete, we have $\lim x_n = u$ for some $u \in X$.

If we now suppose that $d(u, T_1u) \neq 0$ then

$$\begin{aligned}
d^p(x_{2n}, T_1u) &\leqslant H^p(T_2x_{2n-1}, T_1u)\\
&\leqslant \frac{bd^p(u, T_1u)d^s(u, T_2x_{2n-1}) + cd^p(x_{2n-1}, T_2x_{2n-1})d^s(x_{2n-1}, T_1u)}{[\delta(u, T_1u) + \delta(x_{2n-1}, T_2x_{2n-1})]^s}\\
&\leqslant \frac{bd^p(u, T_1u)d^s(u, x_{2n}) + cd^p(x_{2n-1}, x_{2n})d^s(x_{2n-1}, T_1u)}{[d(u, T_1u) + d(x_{2n-1}, x_{2n})]^s}
\end{aligned}$$

and on letting n tend to infinity we have $d(u, T_1u) \leqslant 0$. It follows that $d(u, T_1u) = 0$. Since T_1u is closed, this shows that $u \in T_1u$. By Lemma 1 we conclude that $u \in T_2u$ and $F(T_1) = F(T_2)$ holds.

If $T_1 = T_2$ we have the following theorem.

Theorem 2. Let (X, d) be a complete metric space and let $T: X \to \mathrm{CB}(X)$ be a multifunction with $p \geqslant 1$, $s \geqslant 1$, $b \geqslant 0$, $c \geqslant 0$, $bc < 1$, and if

$$H^p(Tx, Ty) \leqslant \frac{bd^p(x, Tx)d^s(x, Ty) + cd^p(y, Ty)d^s(y, Tx)}{[\delta(x, Tx) + \delta(y, Ty)]^s}$$

holds for all $x, y \in X$ for which $\delta(x, Tx) + \delta(y, Ty) \neq 0$, then T has a fixed point.

If T_1 and T_2 are single valued mappings we have the following result.

Theorem 3. Let T_1 and T_2 be two mappings of a complete metric space (X, d) into itself with $p \geqslant 1$, $s \geqslant 1$, $b \geqslant 0$, $c \geqslant 0$, and $bc < 1$ such that for all x, y in X either

$$d^p(T_1x, T_2y) \leqslant \frac{bd^p(x, T_1x)d^s(x, T_2y) + cd^p(y, T_2y)d^s(y, T_1x)}{[d(x, T_1x) + d(y, T_2y)]^s}$$

if $d(x, T_1x) + d(y, T_2y) \neq 0$ or $d(T_1x, T_2y) = 0$ otherwise, then T_1 and T_2 have a unique common fixed point u.

Proof. The existence of a fixed point follows from Theorem 1. Now suppose that T_1 and T_2 have a second fixed point u'. Then $d(u, T_1u) + d(u', T_2u') = 0$ implies $d(T_1u, T_2u') = 0$ and so $u = T_1u$, $u' = T_2u'$ and $T_1u = T_2u'$. Hence T_1 and T_2 have only one common fixed point.

We note that without the extra condition $d(x, T_1x) + d(y, T_2y) = 0$ implies $d(T_1x, T_2y) = 0$ the common fixed point is not necessarily unique (Fisher 1980).

Remark. If $p = 1$ and $s = 1$ then we obtain Theorem 1 of Fisher (1980).

FIXED POINT THEOREMS FOR A SEQUENCE OF MULTIFUNCTIONS

Theorem 4. Let (X, d) be a complete metric space and let $\{T_n\}_{n \in N}$ be a sequence of multifunctions of X into $\mathrm{CB}(X)$ such that

$$H^pY_1x, T_ny) \leqslant \frac{bd^p(x, T_1x)d^s(x, T_ny) + cd^p(y, T_ny)d^s(y, T_1x)}{[\delta(x, T_1x) + \delta(y, T_ny)]^s} \tag{5}$$

holds for all x, y in X for which $\delta(x, T_1x) + \delta(y, T_ny) \neq 0$, where $n \geqslant 2$, $p \geqslant 1$, $s \geqslant 1$, $b \geqslant 0$, $c \geqslant 0$ and $bc < 1$, then $\{T_n\}_{n \in N}$ has a common fixed point and $F(T_1) = F(T_n)$.

A proof can be obtained by using Theorem 1 and Lemma 1.

Let X be a non-empty set with two metrics e and d on X and $f: X \to X$ a single valued mapping. For such mappings Maia (1968) proved a fixed point theorem which was generalized in many directions by Iséki (1975), Rus (1977a, b), Singh (1978) and others. Popa (1982, 1983, 1984) gave some generalizations of Maia's theorem for multifunctions.

Now we prove a fixed point theorem for a sequence of multifunctions in a set with two metrics.

Theorem 5. Let X be a metric space with two metrics e and d. If X satisfies the following conditions:

(1) $e(x, y) \leqslant d(x, y)$, $\forall x, y \in X$,

(2) X is complete with respect to e,

(3) two multifunctions $T_1, T_2: X \to X$ are punctually closed and punctually bounded with respect to both metrics,

(4) T_1 or T_2 is u.s.c. with respect to e,

(5) the inequality (3) holds for all x, y in X for which $\delta(x, T_1x) + \delta(y, T_2y) \neq 0$ where $p \geqslant 1, s \geqslant 1, b \geqslant 0, c \geqslant 0$ and $bc < 1$, then T_1 and T_2 have a common fixed point and $F(T_1) = F(T_2)$.

Proof. As in the proof of Theorem 1, for any $x_0 \in X$ we can construct a Cauchy sequence with respect to $d\{x_n\}$ such that $x_{2n+1} \in T_1x_{2n}$. Therefore, $e \leqslant d$, $\{x_n\}$ is a Cauchy sequence with respect to e and since X is complete with respect to e, $x_n \to x$. As T_1 is u.s.c. it follows from Theorem 4 (Popa (1982)) that it has a closed graph. From $x_{2n+1} \in T_1x_{2n}$ it follows that $x \in T_1x$ and from Lemma 1 it follows that $F(T_1) = F(T_2)$.

Theorem 6. Let X be a metric space with two metrics e and d. If X satisfies the following conditions:

(1) The sequence of multifunctions $\{T_n\}_{n\in N}$ is formed by punctually closed and punctually bounded multifunctions with respect to both metrics,

(2) e, d and T_1 satisfy conditions (1), (2) and (4) of Theorem 5,

(3) the inequality (5) holds for all x, y in X for which $\delta(x, T_1x) + \delta(y, T_ny) \neq 0$, where $n \geqslant 2$, $p \geqslant 1$, $s \geqslant 1$, $b \geqslant 0$, $c \geqslant 0$ and $bc < 1$, then $\{T_n\}_{n\in N}$ has a common fixed point and $F(T_1) = F(T_n)$.

A proof can be obtained from Theorem 5 and Lemma 1.

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مبرهنات النقطة الثابتة لدوال كثيرة القيم
تحقق متباينة نسبية

ڤاليريو پوپا
قسم الرياضيات ، المعهد العالي للتربية ،
٥٥٠٠ باشو ، رومانيا

خلاصة

لقد أمكن اثبات مبرهنتي نقطة ثابتة لتتابع دوال كثيرة القيم تحقق متباينة نسبية والتي تعمم المبرهنة الأولى من فيشر (١٩٨٠) .