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Nucleus of nearly Kaehler manifolds

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ABSTRACT

It is known that every Kaehler manifold is nearly Kaehler manifold; however, the converse is not always true. In fact the nearly Kaehler condition is much weaker than the Kaehler one. In this paper we study the Kaehler submanifolds of the nearly Kaehler mainifold $(\bar{M}, J)$ which originated as integral submanifolds of the distribution defined at each point $m \in \bar{M}$ by

$$D_m = \{X \in T_m \bar{M} : (\bar{\nabla}_X J)(Y) = 0, \quad Y \in T_m \bar{M}\},$$

and they are named *nuclei* of nearly Kaehler manifold $\bar{M}$.

1. INTRODUCTION

Let $\bar{M}$ be a nearly Kaehler manifold. Then it is an almost hermitian manifold $(\bar{M}, J, g)$, whose almost complex structure J is killing, that is

$$(\bar{\nabla}_X J)(Y) + (\bar{\nabla}_Y J)(X) = 0,\ X,\ Y \in \chi(\bar{M}),$$

where $\bar{\nabla}$ is the Riemannian connection and $\chi(\bar{M})$ is the lie-algebra of vector fields on $\bar{M}$.

It is known that every Kaehler manifold is nearly Kaehler manifold but the converse is not always true. For example, S^6 with standard almost complex structure is nearly Kaehler but not Kaehler. Therefore it becomes interesting to study which submanifolds of a nearly Kaehler manifold carry the induced Kaehler structure. In this paper we pick a simple one introduced by Gray (1970).

Gray (1970) considers a nearly Kaehler manifold $\bar{M}$ and the distribution defined by

$$D_m = \{X \in T_m \bar{M} : (\bar{\nabla}_X J)(Y) = 0, \quad Y \in T_m \bar{M}\},$$

where $T_m \bar{M}$ is the tangent space of $\bar{M}$ at $m \in \bar{M}$. He has proved that the above distribution is integrable and its integral submanifolds are Kaehler manifolds. We call any one of these integral submanifolds as *nucleus* of the nearly Kaehler manifold $\bar{M}$.

The Riemannian connection $\bar{\nabla}$ on $\bar{M}$, induces Riemannian connection ∇ on the nucleus M and the Riemannian connection $\nabla^{\perp}$ in the normal bundle ν of M and they are related by the following formulae:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.1}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{1.2}$$

$X, Y \in \chi(M)$, $N \in \nu$, where $h(X, Y)$ and $A_N X$ are second fundamental forms satisfying $g(A_N X, Y) = g(h(X, Y), N)$ and ν is the normal bundle of M in $\bar{M}$.

According to Chen (1973), the curvature tensors R and $\bar{R}$ of M and $\bar{M}$ respectively are related by

$$\bar{R}(X, Y; Z, W) = R(X, Y; Z, W) - g(h(X, W), h(Y, Z)) + g(h(Y, W), h(X, Z)) \tag{1.3}$$

and

$$\bar{R}(X, Y; N, N') = R^{\perp}(X, Y; N, N') - g([A_N, A_{N'}](X), Y), \tag{1.4}$$

where $R^{\perp}$ is the curvature tensor corresponding to $\nabla^{\perp}$.

If the trace of the second fundamental form h is zero, then M is said to be minimal.

In nearly Kaehler manifold $\bar{M}$ the following equations are well known (Gray 1970):

$$\bar{R}(X, Y; Z, W) = \bar{R}(JX, JY; JZ, JW) \tag{1.5}$$

and

$$\bar{R}(X, Y; Z, W) = \bar{R}(JX, JY; Z, W) + g((\bar{\nabla}_X J)(Y), (\bar{\nabla}_Z J)(W)) \tag{1.6}$$

$$\bar{\mathrm{R}}\mathrm{ic}(X, Y) = \sum_{i=1}^{2n} [R(X, JY; e_1, Je_i) + 2g((\bar{\nabla}_X J)(e_i), (\bar{\nabla}_Y J)(e_i))], \tag{1.7}$$

where $\bar{\mathrm{R}}\mathrm{ic}(X, Y)$ is the Ricci curvature of $\bar{M}$, $\dim \bar{M} = 2n$ and $\{e_i, \ldots, e_{2n}\}$ is an orthonormal frame.

2. NUCLEUS OF NEARLY KAEHLER MANIFOLD

Let (M, J) be the nucleus of nearly Kaehler manifold $(\bar{M}, J)$. It is assumed that $\dim M = 2p$ and $\dim \bar{M} = 2n$. From the definition of M it follows that the tangent space of M is invariant under J and so is the normal space at each point of M.

Proposition 2.1 The nucleus M of the nearly Kaehler manifold $\bar{M}$ is a minimal submanifold.

The proof follows directly from definition of $\bar{M}$ and Eqn (1.1). In fact we get

$$h(X, JY) = h(JX, Y) = Jh(X, Y). \tag{2.1}$$

From this equation, it easily follows that

$$A_{JN} = JA_N \quad \text{and} \quad A_N J = -JA_N. \tag{2.2}$$

A normal section $N \neq 0$ is said to be parallel if $\nabla_X^{\perp} N = 0$ for $X \in \chi(M)$. Now we prove

Lemma 2.1. If $N \neq 0$ is a parallel section in the normal bundle of the nucleus M of nearly Kaehler manifold $\bar{M}$, then so is JN.

Proof. Let $N \neq 0$ be a parallel normal section. Then for each $X \in \chi(M)$, we have

$$(\bar{\nabla}_X J)(N) + (\bar{\nabla}_{JX} J)(JN) = 0, \tag{2.3}$$

which is simply the consequence of definition of nearly Kaehler manifold. This gives

$$\nabla^{\perp}_X JN = J(\nabla^{\perp}_{JX} JN). \tag{2.4}$$

Let $\{N_1, \ldots, N_s, JN_1, \ldots, JN_s\}$ be a local orthonormal frame in v. Suppose that $\nabla^{\perp}_X N_1 = 0$. We want to show that $\nabla^{\perp}_X JN_1 = 0$.

Now

$$g(JN_1, N_1) = 0 \text{ and } \nabla^{\perp}_X N_1 = 0 \Rightarrow g(\nabla^{\perp}_X JN_1, N_1) = 0. \tag{2.5}$$

Also, $g(N_1, N_j) = 0, j = 2, \ldots, s$ and $\nabla^{\perp}_X N_1 = 0$ gives $g(N_1, \nabla^{\perp}_X N_j) = 0$.

Using (2.4) in the form $\nabla^{\perp}_X N = J(\nabla^{\perp}_{JX} N)$, we get

$$g(N_1, J(\nabla^{\perp}_{JX} N_j)) = 0,$$

$$\Rightarrow g(\nabla^{\perp}_{JX} N_j, JN_1) = 0, \quad \text{for all } X \in \chi(M)$$

$$\Rightarrow g(\nabla^{\perp}_X N_j, JN_1) = 0. \tag{2.6}$$

Now, using (2.6) and $g(JN_1, N_j) = 0, j = 2, \ldots, s$, we get

$$g(\nabla^{\perp}_X JN_1, N_j) = 0, j = 2, \ldots, s. \tag{2.7}$$

From (2.5), (2.6) and (2.7) we get

$$\nabla^{\perp}_X JN_1 \perp N_i, i = 1, \ldots, s. \tag{2.8}$$

Thus

$$\nabla^{\perp}_X JN_1 = \sum_{\alpha=1}^{s} \alpha_i JN_i.$$

Making use of (2.4) we have

$$\nabla^{\perp}_{JX} JN_1 = \sum_{i=1}^{s} \alpha_i N_i. \tag{2.9}$$

Also from (2.8) it follows that $\nabla^{\perp}_{JX} JN_1 \perp N_i$. Making use of it in (2.9) we get

$$\alpha_i = 0, i = 1, \ldots, s. \quad \text{Hence } \nabla^{\perp}_{JX} JN_1 = 0, \text{ i.e., } \nabla^{\perp}_X JN_1 = 0.$$

3. FLAT NORMAL CONNECTION

The normal connection $\nabla^{\perp}$ is said to be flat if $R^{\perp} = 0$. In this section we shall investigate the impact of flatness of the normal connection on the geometry of the nucleus, as well as the conditions under which the normal connection is flat.

The following theorems generalise the results for Kaehler submanifold of a Kaehler manifold (Chen & Iue 1975).

Theorem 3.1 If M is the nucleus of the nearly Káehler manifold $\bar{M}$ and the normal connection is flat, then the Ricci tensors Ric and $\bar{\text{R}}\text{ic}$ of M and $\bar{M}$ respectively are equal, i.e.

$$\mathrm{Ric}(X, Y) = \bar{\mathrm{R}}\mathrm{ic}\,(X, Y),\ X,\ Y \in \chi(M).$$

Proof. Since the normal connection is flat, there exists locally $2n - 2p$ mutually orthogonal parallel vector fields $N_1, \ldots, N_{2n-2p}$ (cf. Chen 1973, Prop. 1.1, p. 99). Taking into consideration Lemma 2.1, we see that $\{N_1, \ldots, N_{n-p}, JN_1, \ldots, JN_{n-p}\}$ works as frame of orthonormal parallel sections in the normal bundle. Now taking account of the minimality of M we get from (1.3)

$$\mathrm{Ric}(X, Y) = \bar{\mathrm{R}}\mathrm{ic}(X, Y) - \sum_{\alpha=1}^{n-p}[\bar{R}(N_\alpha, X; Y, N_\alpha) + \bar{R}(JN_\alpha, X; Y, JN_\alpha)] - \sum_{i=1}^{2p} g(h(e_i, X), h(e_i, Y)). \tag{3.1}$$

On the other hand since N_α's are parallel we have from (1.4) and (2.2).

$$\bar{R}(X, Y; N_\alpha, JN_\alpha) = 2g(JA_\alpha^2(X), Y). \tag{3.2}$$

The Bianchi identity gives

$$\bar{R}(X, JY; N_\alpha, JN_\alpha) = -\bar{R}(JN_\alpha, X; N_\alpha, JY) - \bar{R}(N_\alpha, X; JY, JN_\alpha).$$

Making use of (1.6) in the above equation we get

$$\bar{R}(X, JY; N_\alpha, JN_\alpha) = -\bar{R}(JN_\alpha, X; Y, JN_\alpha) + g((\bar{\nabla}_{JN_\alpha} J)(X), (\bar{\nabla}_Y J)(N_\alpha)) - \bar{R}(N_\alpha, X; Y, N_\alpha) + g((\bar{\nabla}_{N_\alpha} J)(X), (\bar{\nabla}_Y J)(N_\alpha)).$$

However, as N_α and JN_α are parallel we have by (2.2)

$$(\bar{\nabla}_X J)(N_\alpha) = 0 \text{ and } (\bar{\nabla}_{N_\alpha} J)(X) = -(\bar{\nabla}_X J)(N_\alpha) = 0.$$

Hence

$$\bar{R}(X, JY; N_\alpha, JN_\alpha) = -[\bar{R}(JN_\alpha, X, Y; JN_\alpha) + \bar{R}(N_\alpha, X; Y, N_\alpha)]. \tag{3.3}$$

Also

$$\sum_{i=1}^{2p} g(h(e_i, X), h(e_i, Y)) = 2 \sum_{i=1}^{n-p} g(A_{N_\alpha}^2(X), Y). \tag{3.4}$$

Making use of (3.2), (3.3) and (3.4) in (3.1), we get the result.

Finally, we discuss the first Chern classes of the nucleus of nearly Kaehler manifold M. Gray (1970) has shown that the first Chern class $c_1(T\bar{M})$ of $\bar{M}$ is represented by

$$\bar{\gamma}_1(X, Y) = \frac{1}{2\pi}\sum_{i=1}^{n}[\bar{R}(X, Y; e_i, Je_i) = \tfrac{1}{2}g(\bar{\nabla}_X J)(e_i), J((\bar{\nabla}_Y J)(e_i))]. \tag{3.5}$$

Making use of (1.7) in this equation we get

$$\bar{\gamma}_1(X, Y) = \frac{1}{2\pi}\,\bar{\mathrm{R}}\mathrm{ic}\,(JX, Y) + 2\sum_{i=1}^{n} g((\bar{\nabla}_{JX} J)(e_i), (\bar{\nabla}_Y J)(e_i)) - \frac{1}{4\pi}\sum_{i=1}^{n} g((\bar{\nabla}_X J)(e_i), J((\bar{\nabla}_Y J)(e_i)). \tag{3.6}$$

Since the nucleus M is Kaehler manifold, its first Chern class $c_1(TM)$ is represented by

$$\gamma_1(X, Y) = \frac{1}{4\pi} \operatorname{Ric}(JX, Y). \tag{3.7}$$

Theorem 3.2. Let M be the nucleus of compact nearly Kaehler manifold $\bar{M}$. If the normal connection is flat, then $c_1(TM) = c_1(v)$, where TM and v are respectively the tangent and normal bundle of M.

Proof: Using the definition of M in (3.6) and combining this with Theorem 3.1 and Eqn (3.7), we get

$$c_1\left(T\bar{M}\Big|_M\right) = 2c_1(TM). \tag{3.8}$$

Also, using $T\bar{M}|_M = TM \oplus v$, we get

$$c_1\left(T\bar{M}\Big|_M\right) = c_1(TM) + c_1(v). \tag{3.9}$$

From (3.8) and (3.9), the theorem follows.

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خلاصة

من المعلوم أن كل ذي طيات كيلر هو ذو طيات كيلر التقريبي ، لكن العكس غير صحيح دائما . ففي الحقيقة ان شرط كيلر التقريبي أضعف كثيرا من شرط كيلر . وفي هذا البحث ندرس ذوات طيات كيلر التقريبية الفرعية التي نشأت كذوات طيات تكاملية فرعية لتوزيع معـرف في نقطة بصيغة معينة ، وهي تسمى جويزات ذي طيات كيلر التقريبية .