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A note on a theorem of Mazhar

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ABSTRACT

In this paper a theorem on $|\bar{N}, p_n|_k$ summability methods, which generalizes a theorem of Mazhar (1977) on $|C, 1|_k$ summability methods, has been proved.

1. INTRODUCTION

A sequence (b_n) is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geqslant -\delta_n$, where (δ_n) is a sequence of positive numbers (Boas 1965). Let $\sum a_n$ be a given infinite series with s_n as its n-th partial sums. We denote by t_n the n-th $(C, 1)$ mean of the sequence (na_n). The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geqslant 1$, if, according to Flett (1957)

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geqslant 1). \tag{2}$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (u_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if, according to Bor (1985)

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{4}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), then $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|\bar{N}, p_n|$) summability.

2. MAZHAR'S THEOREM

Mazhar (1977) proved the following theorem:

Theorem A. Let $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum n\delta_n \log n < \infty$, $\sum A_n \log n$ is convergent and $|\Delta\lambda_n| \leqslant |A_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(\log m) \quad \text{as } m \to \infty, \tag{5}$$

then the series $\sum a_n \lambda_n$ is summable $|C, 1|_k$, $k \geqslant 1$.

3. GENERALIZATION OF THEOREM A

The object of this paper is to generalize Theorem A for $|\bar{N}, p_n|_k$ summability in the form of the following theorem.

Theorem. Let $\lambda_n \to 0$ as $n \to \infty$ and let the sequence (p_n) be such that $np_n = O(P_n)$ and $1/p_n = O(n)$ as $n \to \infty$. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum nP_n \delta_n < \infty$, $\sum A_n P_n$ is convergent and $|\Delta\lambda_n| \leqslant |A_n|$ for all n. If

$$\sum_{n=1}^{m} p_n |t_n|^k = O(P_m) \quad \text{as } m \to \infty, \tag{6}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

4. SOME LEMMAS

We need the following lemmas:

Lemma 1. Under the conditions of the theorem we have

$$P_n |\lambda_n| = O(1) \quad \text{as } n \to \infty. \tag{7}$$

Proof.

$$P_n |\lambda_n| = P_n \left| \sum_{v=n}^{\infty} \Delta\lambda_v \right| \leqslant P_n \sum_{v=n}^{\infty} |\Delta\lambda_v| \leq \sum_{v=n}^{\infty} P_v |\Delta\lambda_v| \leqslant \sum_{v=1}^{\infty} P_v |A_v| < \infty.$$

Hence $P_n |\lambda_n| = O(1)$ as $n \to \infty$.

Lemma 2 (Varshney 1981). Let the sequence (p_n) be such that $np_n = O(P_n)$. If (A_n) is a δ-quasi-monotone sequence with $\sum nP_n \delta_n < \infty$ and $\sum A_n P_n$ is convergent, then

$$mA_m P_m = O(1) \quad \text{as } m \to \infty$$

$$\sum_{n=1}^{\infty} nP_n |\Delta A_n| < \infty. \tag{8}$$

5. PROOF OF THE THEOREM

Let (T_n) be the sequence of $(\bar{N}, p_n)$ means of the series $\sum a_n \lambda_n$. By definition

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{9}$$

Then, for $n \geqslant 1$

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} v a_v. \tag{10}$$

Applying Abel's transformation to the right hand side of (10), we have

$$\begin{aligned}T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \Delta(P_{v-1}\lambda_v/v) \sum_{r=1}^{v} r a_r + \frac{p_n \lambda_n}{n P_n} \sum_{v=1}^{n} v a_v \\ &= \frac{(n+1) p_n t_n \lambda_n}{n P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v} \\ &= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}, \text{ say.}\end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{11}$$

Firstly we have

$$\begin{aligned}\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,1}|^k &= O(1) \sum_{n=1}^{m} p_n |t_n|^k |\lambda_n| |\lambda_n|^{k-1} \\ &= O(1) \sum_{n=1}^{m} |\lambda_n| p_n |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} p_v |t_v|^k + O(1)|\lambda_m| \sum_{n=1}^{m} p_n |t_n|^k \\ &= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| P_n + O(1)|\lambda_m| P_m \\ &= O(1) \sum_{n=1}^{m-1} |A_n| P_n + O(1)|\lambda_m| P_m = O(1)\end{aligned}$$

as $m \to \infty$, by the hypotheses and Lemma 1.

Now, applying Hölder's inequality, as in $T_{n,1}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \right|^k \\ &= O(1) \sum_{n=2}^{m+1} \frac{1}{P_n P_{n-1}} \\ &\quad \times \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^{k-1} |\lambda_v| \frac{1}{P_v} \\ &= O(1) \sum_{v=1}^{m} |\lambda_v| p_v |t_v|^k = O(1) \quad \text{as } m \to \infty.\end{aligned}$$

Again

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} P_v \Delta\lambda_v t_v \frac{v+1}{v} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v |A_v| |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$\times \left\{ \sum_{v=1}^{n-1} P_v |A_v| |t_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v |A_v| \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} P_v |A_v| |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |A_v| |t_v|^k$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{p_v} |A_v| p_v |t_v|^k.$$

Since $\frac{1}{p_v} = O(v)$, by hypothesis, and $P_v < P_{v+1}$, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k = O(1) \sum_{v=1}^{m} v |A_v| p_v |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta(v |A_v|) \sum_{r=1}^{v} p_r |t_r|^k$$

$$+ O(1) m |A_m| \sum_{v=1}^{m} p_v |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} v |\Delta A_v| P_v + O(1) \sum_{v=1}^{m-1} P_v |A_{v+1}|$$

$$+ O(1) m |A_m| P_m$$

$$= O(1) \sum_{v=1}^{m-1} v |\Delta A_v| P_v + O(1) \sum_{v=1}^{m-1} P_{v+1} |A_{v+1}|$$

$$+ O(1) m |A_m| P_m$$

$$= O(1) \quad \text{as } m \to \infty,$$

by the hypothesis and Lemma 2.

Finally, using the fact that $1/v = O(p_v)$, by hypothesis, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,4}|^k = \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left| \sum_{v=1}^{n-1} P_v \lambda_{v+1} t_v \frac{1}{v} \right|^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n=1} P_v |t_v| p_v |\lambda_{v+1}| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$\times \left\{ \sum_{v=1}^{n-1} (P_v |\lambda_{v+1}|)^k p_v |t_v|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (P_v |\lambda_{v+1}|)^{k-1}$$

$$\times P_v |\lambda_{v+1}| p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} (P_v |\lambda_{v+1}|)^{k-1} |\lambda_{v+1}| p_v |t_v|^k.$$

Since $P_v < P_{v+1}$, we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,4}|^k = O(1) \sum_{v=1}^{m} (P_{v+1} |\lambda_{v+1}|)^{k-1} |\lambda_{v+1}| p_v |t_v|^k$$

$$= O(1) \sum_{v=1}^{m} |\lambda_{v+1}| p_v |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta |\lambda_{v+1}| \sum_{r=1}^{v} p_r |t_r|^k$$

$$+ O(1) |\lambda_{m+1}| \sum_{v=1}^{m} p_v |t_v|^k$$

$$= O(1) \sum_{v=1}^{m=1} |\Delta\lambda_{v+1}| P_v + O(1) |\lambda_{m+1}| P_m$$

$$= O(1) \sum_{v=1}^{m-1} |A_{v+1}| P_{v+1} + O(1) |\lambda_{m+1}| P_{m+1}$$

$$= O(1) \quad \text{as } m \to \infty,$$

by virtue of the hypotheses and Lemma 1. Therefore, we get

$$\sum_{n=1}^{m} (P_n/p_n)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2, 3, 4.$$

This completes the proof of the theorem.

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خلاصة

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