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Singular perturbation for a non-autonomous second order differential equation with retarded argument

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ABSTRACT

This paper studies the existence and the uniqueness of a periodic solution for a singularly perturbed nonlinear non-autonomous second order differential equation with retarded argument. The perturbation has a small parameter and the retardation is assumed to be constant.

INTRODUCTION

We consider the differential equation

$$\varepsilon\ddot{u}(t) + g(t, u(t), \dot{u}(t), u(t-\tau), \dot{u}(t-\tau), \varepsilon) = 0, \tag{1}$$

where g is a real-valued function, $\varepsilon(\varepsilon > 0)$ is a small parameter, the retardation $\tau(\tau > 0)$ is constant and $\dot{} \doteq \frac{d}{dt}$. Also, assume that Eqn (1), when $\varepsilon = 0$, admits a periodic solution $\omega(t)$ with period $T(T > \tau)$. Periodic solutions for unperturbed and perturbed nonlinear second order differential equations have been studied (see, e.g. Levinson & Smith (1942), Cesari (1971), Farkas & Farkas (1972), Yoshizawa (1975) and El-Owaidy (1975).

In the following, we study the problem of existence of a unique periodic solution, with period T, for Eqn (1). This problem was studied generally, for equations without retardation, by El-Nahhas (1985).

ASSUMPTIONS

For Eqn (1) assume the following:

(i) g is analytic with respect to its arguments on some region R such that the point

$$(t, \omega(t), \dot{\omega}(t), \omega(t-\tau), \dot{\omega}(t-\tau), 0) \in R.$$

(ii) g is periodic in t with period T.

(iii) $g_{\dot{u}}(t), g_{\dot{u}_\tau}(t), h(t) > K$ (constant) > 0 for all values of t, where

$$h(t) = g_{\dot{u}}^{-1}(t)g_u(t) + g_{\dot{u}_\tau}^{-1}(t)g_{u_\tau}(t),$$

$$g_{u^{(i)}}(t) \equiv g_{u^{(i)}(t)}(t, \omega(t), \dot{\omega}(t), \omega(t-\tau), \dot{\omega}(t-\tau), 0), \qquad i = 0, 1,$$

$$g_{u_\tau^{(i)}}(t) \equiv g_{u^{(i)}(t-\tau)}(t, \omega(t), \dot{\omega}(t), \omega(t-\tau), \dot{\omega}(t-\tau), 0), \qquad i = 0, 1.$$

(iv) The homogeneous linear system

$$\dot{y}(t) = y_2(t) - h(t)y_1(t),$$

$$\varepsilon\dot{y}_2(t) = -g_{\dot{u}}(t)y_2(t) - g_{\dot{u}_\tau}(t)y_2(t-\tau) - l(t)y_1(t) - l_\tau(t)y_1(t-\tau)$$

where

$$l(t) = g_u(t) - h(t)g_{\dot{u}}(t),$$

$$l_\tau(t) = g_{u_\tau}(t) - h(t-\tau)g_{\dot{u}_\tau}(t),$$

have no periodic solutions with period T.

Theorem. If the above assumptions, (i)–(iv), are satisfied, then Eqn (1) admits a unique periodic solution, with period T, which tends to $\omega(t)$ as $\varepsilon \to 0$.

Proof. Introducing $x_1(t) = u(t)$, $x_2(t) = \dot{u}(t)$, Eqn (1) will be equivalent to the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \varepsilon\dot{x}_2(t) &= -g(t, x_1(t), x_2(t), x_{1\tau}(t), x_{2\tau}(t), \varepsilon) \end{aligned} \tag{2}$$

where

$$x_{i\tau}(t) \equiv x_i(t-\tau), \qquad i = 1, 2.$$

When $\varepsilon = 0$, System (2) will admit the periodic solution

$$P(t) = \mathrm{col}[\,|\,\omega(t), \dot{\omega}(t)].$$

Since Assumptions (i) and (iii) emphasize that $g_{x_2}(t)$,

$$g_{x_{2\tau}}(t) \neq 0 \forall t \text{ and } g_{x_i}(t), g_{x_{i\tau}}(t) \qquad (i = 1, 2)$$

all exist, then we can perform the transformations

$$\begin{aligned} y_1(t) &= x_1(t) - \omega(t), \\ y_2(t) &= x_2(t) - \dot{\omega}(t) - q(t), \\ q(t) &= -y_1(t)h(t), \\ h(t) &= g_{x_2}^{-1}(t)g_{x_1}(t) + g_{x_2}^{-1}(t)g_{x_{1\tau}}(t) \end{aligned} \tag{3}$$

which transforms (2) to the form

$$\begin{aligned} &\dot{y}_1(t) = y_2(t) + q(t), \\ &\varepsilon\dot{y}_2(t) = -g(t, y_1(t) + \omega(t), y_2(t) + \dot{\omega}(t) + q(t), \\ &\quad y_{1\tau}(t) + \omega_\tau(t), y_{2\tau}(t) + \dot{\omega}_\tau(t) + q_\tau(t), \varepsilon) - \varepsilon\ddot{\omega}(t) - \varepsilon\dot{q}(t), \\ &y_{i\tau}(t) \equiv y_i(t-\tau) \quad (i = 1, 2), \qquad \omega_\tau^{(i)}(t) \equiv \omega^{(i)}(t-\tau) \quad (i = 1, 0), \\ &q_\tau(t) \equiv q(t-\tau). \end{aligned} \tag{4}$$

According to Transformation (3) and Assumption (ii), we can easily deduce:

Remark 1. If $y_1(t), y_2(t) \to 0$, then $x_1(t) \to \omega(t)$, $x_2(t) \to \dot{\omega}(t)$.

Remark 2. If System (4) admits a periodic solution with period T, then System (2), also, admits periodic solution with the same period.

Also, as a consequence of Assumption (i), the function $g(t, x_1(t), x_2(t), x_{1\tau}(t), x_{2\tau}(t), \varepsilon)$ can be expanded at the point $(t, \omega(t), \dot{\omega}(t), \omega_\tau(t), \dot{\omega}_\tau(t), 0)$

$$g(t, x_1, x_2, x_{1\tau}, x_{2\tau}, \varepsilon) = g(t, \omega, \dot{\omega}, \omega_\tau, \dot{\omega}_\tau, 0) + \sum_{i=1}^{2} [g_{x_i}(t)(x_i - \omega^{(i-1)}) + g_{x_{i\tau}}(t)(x_{i\tau} - \omega^{(i-1)})] + \varepsilon g_\varepsilon(t) + f(t, x_1, x_2, x_{1\tau}, x_{2\tau}, \varepsilon),$$

$$g_\varepsilon(t) \equiv g_\varepsilon(t, \omega(t), \dot{\omega}(t), \omega_\tau(t), \dot{\omega}_\tau(t), 0),$$

$$f(t, x_1, x_2, x_{1\tau}, x_{2\tau}, \varepsilon) = 0(|x_1|^2 + |x_2|^2 + |x_{1\tau}|^2 + |x_{2\tau}|^2 + \varepsilon^2).$$

And hence System (4) can be put in the form

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) - h(t)y_1(t), \\ \varepsilon\dot{y}_2(t) &= -g_{x_2}(t)y_2(t) - [g_{x_1}(t) - h(t)g_{x_2}(t)]y_1(t) - g_{x_{2\tau}}(t)y_{2\tau}(t) \\ &\quad - [g_{x_{1\tau}}(t) - h_\tau(t)g_{x_{2\tau}}(t)]y_{1\tau}(t) - \varepsilon[\ddot{\omega}(t) + g_\varepsilon(t)] + \lambda(t, y_1, y_2, y_{1\tau}, y_{2\tau}, \varepsilon), \quad (5) \\ \lambda &= 0(|y_1|^2 + |y_2|^2 + |y_{1\tau}|^2 + |y_{2\tau}|^2 + \varepsilon^2), \\ h_\tau(t) &\equiv h(t - \tau). \end{aligned}$$

Now, we shall apply the fixed point method to prove the existence of a unique periodic solution for System (5) which tends to zero as $\varepsilon \to 0$.

Consider the space S of continuous functions $\alpha(t, \varepsilon)$, $\beta(t, \varepsilon)$, which are periodic in t with period T and satisfy

$$|\alpha(t, \varepsilon)|, |\alpha_\tau(t, \varepsilon) < K_1\varepsilon, |\beta(t, \varepsilon)|, |\beta_\tau(t, \varepsilon)| < K_2\varepsilon$$

where K_1, K_2 are arbitrary positive constants and

$$\alpha_\tau(t, \varepsilon) \equiv \alpha(t - \tau, \varepsilon), \qquad \beta_\tau(t, \varepsilon) \equiv \beta(t - \tau, \varepsilon).$$

Also, consider the system

$$\begin{aligned} \dot{y}_1(t) &= y_2(t) - h(t)y_1(t), \\ \varepsilon\dot{y}_2(t) &= -g_{x_2}(t)y_2(t) - g_{x_{2\tau}}(t)y_{2\tau}(t) - l(t)y_1(t) - l_\tau(t)y_{1\tau}(t) - \varepsilon\eta(t) \quad (6) \\ &\quad + \lambda(t, \alpha, \beta, \alpha_\tau, \beta_\tau, \varepsilon) \end{aligned}$$

where

$$\begin{aligned} l(t) &= g_{x_1}(t) - h(t)g_{x_2}(t), \\ l_\tau(t) &= g_{x_{1\tau}}(t) - h_\tau(t)g_{x_{2\tau}}(t), \\ \eta(t) &= \ddot{\omega}(t) + g_\varepsilon(t). \end{aligned}$$

Let Γ be the operator which maps the functions $\alpha(t, \varepsilon)$, $\beta(t, \varepsilon)$ respectively to the unique periodic solution $z_1(t, \varepsilon)$, $z_2(t, \varepsilon)$ of (6). Assumption (iv) of the theorem assures the existence of this solution.

Using the fact that z_1, z_2 is a solution of (6) and Assumptions (i) and (iii), we can prove that

$$|z_1(t, \varepsilon)|, |z_{1\tau}(t, \varepsilon)| < K_1'\varepsilon, |z_2(t, \varepsilon)|, |z_{2\tau}(t, \varepsilon) < K_2' \varepsilon$$

where K_1' and K_2' are positive constants which depend on K_1 and K_2, $z_{i\tau}(t, \varepsilon) \equiv z_i(t - \tau, \varepsilon)$ $(i = 1, 2)$. This implies that Γ maps the space S into itself.

Also, if Γ maps α_1, β_1 to z_1, z_2 and α_1', β_1' to z_1', z_2', it is easy to prove that

$$|z_1 - z_1'| + |z_2 - z_2'| + |z_{1\tau} - z_{1\tau}'| + |z_{2\tau} - z_{2\tau}'| \leqslant \eta\gamma,$$

$$\gamma = |\alpha_1 - \alpha_1'| + |\beta_1 - \beta_1'| + |\alpha_{1\tau} - \alpha_{1\tau}'| + |\beta_{1\tau} - \beta_{1\tau}'|,$$

where $0 < \eta < 1$, and consequently, Γ is a contraction. Thus, Γ admits a unique fixed point, and this means that System (4) admits a unique periodic solution with period T which tends to zero as $\varepsilon \to 0$. According to Remarks 1 and 2, System (2) admits a unique periodic solution with period T which tends to $P(t)$ as $\varepsilon \to 0$, and hence Eqn (1) admits a unique periodic solution which tends to $\omega(t)$ as $\varepsilon \to 0$. This completes the proof.

Remark. We obtain similar conditions in the case of variable retardation $\tau(t)$, but we must add the following conditions on $\tau(t)$:

(1) $\tau(t) \in C^1$ and $\dot{\tau}(t) \neq 1 \forall t$.
(2) $\tau(t)$ may be periodic in t with period T $(T > \tau(t) \forall t)$.

Example. Consider the equation

$$\varepsilon\ddot{u}(t) + \frac{\tau^2}{2\pi^2}[u(t) + u(t - \tau)] + \frac{\tau}{\pi}[\dot{u}(t) + \dot{u}(t - \tau)] + \frac{\varepsilon\pi^2}{\tau^2}\sin\left(\frac{\pi}{\tau}t\right) = 0, \tag{7}$$

where $\varepsilon(\varepsilon > 0)$ is a small parameter and $\tau = 2\pi$.

Accordant with our assumptions, τ, here, is a positive constant and, when $\varepsilon = 0$, we have the equation

$$\tau[u(t) + u(t - \tau)] + 2\pi[\dot{u}(t) + \dot{u}(t - \tau)] = 0,$$

which has the periodic solution $\omega(t) = \sin\left(\frac{\pi}{\tau}t\right)$ with period $T = 2\tau$ (i.e. $T > \tau$). The function g, for this example, has the form

$$g(t, u(t), \dot{u}(t), u(t - \tau), \dot{u}(t - \tau), \varepsilon) = \frac{\tau^2}{2\pi^2}[u(t) + u(t - \tau)] + \frac{\tau}{\pi}[\dot{u}(t) + \dot{u}(t - \tau)] + \frac{\varepsilon\pi^2}{\tau^2}\sin\left(\frac{\pi}{\tau}t\right).$$

This function is analytic in its arguments on the whole real space and periodic in t, with period T, which coincides with Assumptions (i) and (ii) of the theorem. Also, for $K = 1$, in Assumption (iii) we have

$$g_{\dot{u}}(t) = g_{\dot{u}_\tau}(t) = h(t) = \frac{\tau}{\pi} = 2 > K > 0.$$

For Example (7), the homogeneous linear system of Assumption (iv) of the theorem takes the form

$$\dot{y}_1(t) = y_2(t) - \frac{\tau}{\pi}\, y_1(t),$$

$$\varepsilon \dot{y}_2(t) = -\frac{\tau}{\pi}\,[y_2(t) + y_2(t-\tau)] + \frac{\tau^2}{2\pi^2}\,[y_1(t) + y_1(t-\tau)],$$

or briefly, in vector notation

$$\varepsilon \dot{Y}(t) = AY(t) + BY(t-\tau), \tag{8}$$

where

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{\varepsilon\tau}{\pi} & \varepsilon \\ \dfrac{\tau^2}{2\pi^2} & -\dfrac{\tau}{\pi} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \dfrac{\tau^2}{2\pi^2} & -\dfrac{\tau}{\pi} \end{bmatrix}.$$

Any solution of (8) can be written, $Y(t) = \exp(tC)$, where C is the matrix such that $\exp(tC)$ commutes with both A and B and $\varepsilon C = A + B\exp(-\tau C)$. It is clear that $Y(t)$ is not periodic with period T, and this is coincident with Assumption (iv) of the theorem. On the other hand, for sufficiently small ε, Eqn (7) has the two solutions $u_1(t) = \sin\left(\frac{\pi}{\tau}t\right)$, $u_2(t) = e^{-t} + \sin\left(\frac{\pi}{\tau}t\right)$. One of these solutions, namely $u_1(t)$, is periodic with period T, while the other does not have this property. This, of course, agrees with the result of the theorem.

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الحيود غير العادي لمعادلة تفاضلية
غير ذاتية من الرتبة الثانية ذات متغيرات متأخرة

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خلاصة

تناول البحث دراسة وجود ووحدوية حل دوري لمعادلة تفاضلية محادة لا خطية غير ذاتية من الرتبة الثانية وذات متغيرات متأخرة ، عندما يكون الحيود في المعادلة من النوع غير العادي .