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The Tychonoff Product Theorem for finite spaces and the Boolean Ultrafilter Theorem are equivalent

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ABSTRACT

The (Weak) Tychonoff Product Theorem for finite spaces is shown to be equivalent to the Boolean Ultrafilter Theorem (BUT). Some known equivalents of the BUT are given.

It is well known (Kelley 1950) that the Tychonoff Product Theorem for compact spaces is equivalent (in a reasonable set theory) to the Axiom of Choice. The original motivation for this paper was an immediate observation of the equivalence between the Tychonoff Product Theorem for finite spaces and the Boolean Ultrafilter Theorem (every nontrivial Boolean algebra has an ultrafilter).

The Boolean Ultrafilter Theorem is known (Jech 1973) to be strictly weaker than the AC, but not provable in ZF. We present some, in fact known (Banaschewski 1985, 1986; Banaschewski & Harting 1985; Blass 1987; Jech 1973; Moore 1982), relationships between the BUT and certain frame-theoretic conditions. The equivalence between the (Weak) Tychonoff Product Theorem for finite spaces and the BUT is given.

All unexplained facts concerning frames can be found in Johnstone (1982) or in Paseka & Šmarda (1987). Recall that a frame is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$$

holds for any $a \in L$, $S \subseteq L$.

Frames can be viewed as generalized topological spaces. Frames which are isomorphic to the frame $O(X)$ of all open subsets of a suitable topological space X are called topologies or spatial frames.

Let L be a frame. L is said to be conjunctive if, given $a, b \in L$ with $a < b$, we can find $c \in L$ such that $a \vee c < 1$ (1 is the top element of L), $b \vee c = 1$. L is said to be regular, 0-dimensional respectively (see Johnstone 1982) if

$a = \bigvee\{x \in L : x^* \vee a = 1\}$, $a = \bigvee\{x \in L : x^* \vee x = 1,\ x \leqslant a\}$ respectively, for each $a \in L$; here x^* is the pseudocomplement of x. We say that L is compact if 1 is a compact element, i.e. $\bigvee S = 1$ implies there is $F \subseteq S$, F finite such that $\bigvee F = 1$ for all $S \subseteq L$.

Let us notice that from Johnstone (1982). III 1.2) we have

Lemma 1. Let L be a regular frame, K a compact frame and $f: L \to K$ a surjective frame homomorphism which is dense, i.e. $f(a) = 0$ (0 is the bottom element of L) implies $a = 0$. Then L is isomorphic to K.

We say that an element $1 \neq p \in L$ is prime if $x \wedge y \leqslant p$ implies $x \leqslant p$ or $y \leqslant p$ for all x, $y \in L$. It is easy to check that L is spatial iff any element of L is a meet of primes, i.e. whenever $a < b$ in L there is a prime element $p \in L$ such that $a \leqslant p$ but $p < b \vee p$.

Observation. Let $(f_i : K_i \to L_i)$ be a collection of monomorphisms of frames preserving arbitrary meets. Then the induced morphism $f = \Sigma f_i : \Sigma K_i \to \Sigma L_i$ is a monomorphism again.

Proof. Using standard techniques for sums introduced in Paseka & Šmarda (1987) we obtain the required result.

Proposition 1. The following statements are equivalent:

(i) The sum of finite frames is a spatial frame.
(ii) The Tychonoff Product Theorem for finite topological spaces.
(iii) The Tychonoff Product Theorem for finite discrete spaces.
(iv) The sum of finite Boolean algebras is a spatial frame.

Proof. (i) ⇒ (ii). Let (X_i, T_i) be a collection of finite topological spaces. Clearly, $\Sigma T_i \cong O(\Pi(X_i, T_i))$ is spatial. Now, since the Tychonoff Theorem for sums of compact frames is valid without any use of choice (Johnstone 1982; Kříž 1985) we have that $\Pi(X_i, T_i)$ is a compact topological space.

(ii) ⇒ (iii). It is evident.

(iii) ⇒ (iv). Let (B_i) be a collection of finite Boolean algebras. Then $B_i \cong 2^{X_i}$ for some finite set X_i. Clearly, we have a surjective dense frame morphism (Johnstone 1982) $f: \Sigma B_i \to O(\Pi(X_i, 2^{X_i}))$. The rest follows from Lemma 1.

(iv) ⇒ (i). Let (L_i) be a collection of finite frames. Obviously, we can find a suitable Boolean algebra B_i so that there is a monomorphism $v_i : L_i \to B_i$. Now, by the preceding observation we have an induced frame monomorphism $v : \Sigma K_i \to \Sigma B_i$, i.e. ΣK_i is a spatial frame because any subframe of a spatial frame is spatial again.

Theorem 1. The following are equivalent:

(i) The Boolean Ultrafilter Theorem.
(ii) Every compact conjunctive frame is spatial.
(iii) Every compact regular frame is spatial.

(iv) The sum of compact regular frames is spatial.
(v) The sum of finite Boolean algebras is spatial.
(vi) The sum of 4-element Boolean algebras is spatial.
(vii) Every compact 0-dimensional frame is spatial.

Proof. (i) $\Rightarrow$ (ii). See Banaschewski (1985).
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). It is immediate.
(vi) $\Rightarrow$ (vii). Since any compact 0-dimensional frame is a homomorphic image of a suitable sum of 4-element Boolean algebras, it is isomorphic to some closed homomorphic image of this sum. The rest follows from the fact that any closed homomorphic image of a topology is a topology again.
(vii) $\Rightarrow$ (i). Let B be a nontrivial Boolean algebra, Id(B) the frame of all ideals of B. Evidently, Id(B) is a compact 0-dimensional frame, i.e. there is a maximal ideal in B.

Now, we can state the following well known fact (Jech 1973) that the BUT is equivalent with the Tychonoff Product Theorem for 2-point discrete spaces.

Corollary 1. The following are equivalent:

(i) The Boolean Ultrafilter Theorem.
(ii) The Tychonoff Product Theorem for finite topological spaces.
(iii) The Tychonoff Product Theorem for 2-point discrete topological spaces.

Ward (1962) showed that the following Weak Tychonoff Product Theorem is equivalent to the Axiom of Choice:

The product of any number of copies of a given compact space is compact. Now, we establish the following:

Corollary 3. The following statements are equivalent:

(i) The Boolean Ultrafilter Theorem.
(ii) The Weak Tychonoff Product Theorem for compact sober spaces.
(iii) The Weak Tychonoff Product Theorem for compact Hausdorff spaces.
(iv) The Weak Tychonoff Product Theorem for finite topological spaces.
(v) The Weak Tychonoff Product Theorem for finite discrete topological spaces.

Proof. (i) $\Rightarrow$ (ii). It follows from the fact that the Tychonoff Product Theorem for compact sober spaces is equivalent to the BUT (Banaschewski 1986; Johnstone 1984). The remaining implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) are immediate.

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