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Another version of the operator and scaling-variational methods

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ABSTRACT

A recently developed two-step approach to anharmonic oscillators and a perturbation theory for the same problem, both based on a Bogoliubov transformation, are shown to be equivalent to the scaling-variational and operator methods, respectively.

Hsue & Chern (1984) developed a two-step approach to anharmonic oscillators based on the Bogoliubov transformation

$$b = (1 - t^2)^{-1/2}(a - ta^+), \tag{1a}$$

$$b^+ = (1 - t^2)^{-1/2}(a^+ - ta). \tag{1b}$$

where a^+ and a are the creation and annihilation operators, respectively, defined in terms of the coordinate x and conjugate momentum $p = -i(d/dx)$ as

$$a^+ = 2^{-1/2}(x - ip), \tag{2a}$$

$$a = 2^{-1/2}(x + ip). \tag{2b}$$

These authors obtained quite accurate eigenvalues through diagonalization of the matrix of

$$H = \tfrac{1}{2}(p^2 + x^2) + \lambda x^4 \tag{3}$$

in the basis set of eigenvectors of the number operator b^+b.

The Bogoliubov transformation (1) has recently been applied to perturbation theory by Patnaik (1986) who proposed to split the Hamiltonian operator (3) as

$H = H_0 + H'$, where

$$H_0 = E_0 + \omega b^+ b + \frac{3\lambda}{2\omega^2} b^{+2} b^2 \tag{4}$$

$$H' = \frac{\lambda}{4\omega^2} (b^{+4} + b^4 + 4b^{+3} + 4b^+ b^3), \tag{5}$$

$E_0 = (1 + 3\omega^2)/8\omega$ and $\omega = (1 - t)/(1 + t)$. The eigenvalues of H can be approximately obtained through perturbation theory where H' is considered to be a perturbation (Patnaik 1986). The parameter ω is determined according to the variational principle $dE(\omega)/d\omega = 0$, where $E(\omega) = \langle\Phi | H | \Phi\rangle / \langle\Phi | \Phi\rangle$ and $|\Phi\rangle$ is a generalized coherent state of the form $|\Phi\rangle = \exp[(t/2)a^{+2}] |0\rangle$ (Hsue & Chern 1984; Patnaik 1986). The 'vacuum' state $|0\rangle$ obeys a $|0\rangle = 0$.

It is shown here that none of those procedures is original since the Bogoliubov transformation (1) is equivalent to the scaling transformation (see Fernández & Castro 1983a, b and Gerry & Silverman 1983, 1984 and references therein) that has already been used in connection with the Rayleigh-Ritz variational method (Lu & Nigan 1969; Reid 1970; Balsa *et al.* 1983; Quick & Miller 1985; Gerry 1986) and perturbation theory (Gerry & Silverman 1983, 1984; Feranchuk & Komarov 1982, 1984; Fernández & Castro 1982; Fernández *et al.* 1984, 1985a, b). In the latter case we obtain the operator method (Gerry & Silverman 1983, 1984; Feranchuk & Komarov 1982, 1984; Fernández & Castro 1982, Fernández *et al.* 1984, 1985a, b) that yields highly accurate eigenvalues when the adjustable parameter ω is properly chosen (Fernández & Castro 1982; Fernández *et al.* 1984, 1985a, b).

To prove that the Bogoliubov and scaling transformations are equivalent, let us introduce a new coordinate x' and conjugate momentum p' as follows

$$x' = 2^{-1/2}(b + b^+), \tag{6a}$$

$$p' = 2^{-1/2} i(b^+ - b). \tag{6b}$$

It can be easily verified that $x' = \omega x$ and $p' = \omega^{-1} p$ which is the scaling transformation. Therefore, the old and new coordinates and momenta are related by means of the unitary transformation (Fernández & Castro 1983a, b; Gerry & Silverman 1983, 1984)

$$x' = U^+ x U, \quad p' = U^+ p U \tag{7}$$

where the operator U is defined by

$$U = \exp[\tfrac{1}{2} \ln \omega (a^{+2} - a^2)]. \tag{8}$$

It appears to be preferable to define the ground-state variational ansatz as $|\Phi\rangle = U|0\rangle$ since U preserves the vector norm.

It follows immediately from what was said above that the variational method proposed by Hsue & Chern (1984) is not substantially different from the one discussed by Lu & Nigan (1969), Reid (1970), Balsa *et al.* (1983), Gerry (1986) and Quick & Miller (1985) and that the perturbation theory developed by Patnaik (1986) is equivalent to the operator method (Gerry & Silverman 1983, 1984; Feranchuk & Komarov 1982, 1984; Fernández & Castro 1982; Fernández *et al.* 1984, 1985a, b).

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صور أخرى من المؤثر وطرائق التغيير التدريجي

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خلاصة

لقد أمكن اثبات ان المعالجة الحديثة ذات الخطوتين للمتذبذب التوافقي ونظرية الاهتزازات لنفس المسألة ، والمبنيتين كليهما على تحويل بوجوليوبوف ، هما مكافئتان لطريقتي التغيير التدريجي والمؤثر على التوالي .