J. Univ. Kuwait (Sci.) **17**, 1990

On periodic solutions of the rotational motion of a rigid body

F. M. EL-SABAA

Department of Mathematics, University of Kuwait, P.O. Box 5969, Safat 13060, Kuwait

ABSTRACT

The equation of motion of a heavy rigid body about a fixed point is transformed, using isothermal coordinates on the inertia ellipsoid, to the equations of plane motion of a fictitious material point. The Liapunov's method of holomorphic integral is used to get a family of periodic solutions.

INTRODUCTION

It is well known that the nonlinear differential equations of a rotating heavy rigid body about a fixed point can be solved in three cases namely, Euler's, Lagrange's and Kovalevski's, in which restrictions are imposed only on the mass distribution in the rigid body (Golubev 1953). The solutions in the first two cases can be expressed in terms of elliptic functions, while in the third case, the solution is given by hyper-elliptic functions (Whittaker 1965). Furthermore, particular solutions by quadratures can be obtained under certain restrictions on the initial values of projections of angular velocity for the body on the principal axes of inertia, and the direction cosines of the vertical.

Some other particular solutions have been found in the form of power series involving a small parameter (Andoyer 1923; MaçMillan 1936; Khalimanovich 1953; Toporova 1962). Kolossoff (1903) has solved this problem by reducing the order of the differential equations describing the motion of the rigid body. His results state that, if a heavy rigid body which rotates about a fixed point possesses energy integral and cyclic integral, then by using the isothermal coordinates on the surface of the inertia ellipsoid, the system will be reduced to the plane motion equations of a fictitious material point under the action of potential and gyroscopic forces.

The above mentioned results are well known since 1903, and they were used only by Kharlamov (1963) and Vagner & Demin (1975). Kharlamov (1963) proved that the fourth first integral does not exist in the linear form, while Vagner & Demin (1975) used Kolossoff's results to get the periodic solutions by small parameter's method.

In this paper we will start with the reduced equations of plane motion to establish the family of periodic solutions near the equilibrium points, using Liapunov's (1966) method for holomorphic integral.

1. REDUCTION OF ORDER OF EQUATIONS OF MOTION

Let $OXYZ$ be a fixed coordinate system with origin at a fixed point 0, where Z-coordinate axis is directed vertically upward, and let $Oxyz$ be a moving system of axes directed along the principal axes of inertia for point 0. Let a, b and c be the coordinates of the centre of mass in the moving coordinate system. The Lagrangian function of the rotation of rigid body is

$$L = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) - mg(a\gamma_1 + b\gamma_2 + c\gamma_3)$$

where

$$p = \dot{\psi} \sin\theta \sin\phi + \dot{\theta}\cos\phi$$
$$q = \dot{\psi} \sin\theta \cos\dot{\phi} - \theta \sin\phi$$
$$r = \dot{\psi}\cos\theta + \dot{\phi}$$

are the projections of the angular velocity of the body on the principal axes of inertia; γ_1, γ_2 and γ_3 are direction cosines of the vertical;

$$\gamma_1 = \sin\theta \sin\phi, \qquad \gamma_2 = \sin\theta\cos\phi, \qquad \gamma_3 = \cos\theta;$$

A, B and C are the principal moments of inertia; the angles θ, ϕ and ψ are the Euler's angles defined here with respect to the ascending node, where ψ is a cyclic coordinate (Precession angle), ϕ and θ are the angles of rotation and nutation, respectively.

To write Lagrangian in terms of the generalized coordinates (θ, ϕ, ψ) and generalized velocities $(\dot{\theta}, \dot{\phi}, \dot{\psi})$, we have

$$L = \tfrac{1}{2}[A(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)^2 + B(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)^2 + C(\dot{\psi}\cos\theta + \dot{\phi})^2] - U(\theta, \phi). \tag{1}$$

From Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \qquad (i = 1, 2, 3) \tag{2}$$

we can get the equation

$$\frac{\partial L}{\partial \dot{\psi}} = n, \qquad \text{where } n \text{ is a constant.}$$

The last equation gives $\dot{\psi}$ in terms of the other variables

$$\dot{\psi} = \frac{n - (A - B)\dot{\theta}\sin\theta\sin\phi\cos\phi - C\dot{\phi}\cos\theta}{D} \tag{3}$$

where D will be defined below.

The Routh's function in the system is

$$R = L - n\dot{\psi}$$

which can be written as a function of $(\theta, \phi, \dot{\theta}, \dot{\phi}; n)$ as follows

$$R = R_2 + R_1 + R_0 \tag{4}$$

where

$$R_2 = \frac{D}{2}\Big\{C(A\sin^2\phi + B\cos^2\phi)\sin^2\theta\dot{\phi}^2$$
$$- 2(A - B)\sin\theta\sin\phi\cos\theta\cos\phi\dot{\theta}\dot{\phi}$$
$$+\left[\frac{1}{D}(A\cos^2\phi + B\sin^2\phi) - (A - B)^2\sin^2\theta\sin^2\phi\cos^2\phi\right]\dot{\theta}^2\Big\}, \quad (5)$$

$$R_1 = nD[(A - B)\sin\theta\sin\phi\cos\phi\dot{\theta} + C\cos\theta\dot{\phi}], \quad (6)$$

$$R_0 = U(\theta, \phi) - \frac{n^2}{2}D \quad (7)$$

and

$$D^{-1} = A\sin^2\phi\sin^2\theta + B\cos^2\phi\sin^2\theta + C\cos^2\theta.$$

Introduce the new variables, X, Y, Z such that

$$X = \frac{1}{\sqrt{A}}\sin\theta\sin\phi, \qquad Y = \frac{1}{\sqrt{B}}\sin\theta\cos\phi, \qquad Z = \frac{1}{\sqrt{C}}\cos\theta$$

then, Expressions (5), (6) and (7) become

$$R_2 = \tfrac{1}{2}ABCD(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2), \quad (5)'$$

$$R_1 = \frac{nD\sqrt{ABC}}{AX^2 + BY^2}[C(\dot{X}Y - \dot{Y}X)Z - \dot{Z}(A - B)XY], \quad (6)'$$

$$R_0 = mg(a\sqrt{A}X + b\sqrt{B}Y + c\sqrt{C}Z) - \frac{n^2}{2}D. \quad (7)'$$

The Lagrange's equations for non-cyclic coordinate become

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i}\right) = 0, \qquad (i = 1, 2), \quad (8)$$

which is a system of two degrees of freedom.

The energy integral of the system can be found from

$$H \equiv \sum_{i=1}^{2}\frac{\partial R}{\partial \dot{q}_i}\dot{q}_i - R = h \quad (9)$$

where R is independent function of t.

Vagner & Demin (1975) expressed the Euler's angles θ, ϕ in terms of elliptic functions as follows:

$$\cos\theta = \frac{dn(u, k)cn(s, k')}{dn(s, k')} = \sqrt{c}Z,$$

$$\tan\phi = \frac{sn(u, k)}{cn(u, k)sn(s, k')} = \sqrt{\frac{A}{B}}\frac{X}{Y} \quad (10)$$

where u and s are the isothermal coordinates on the surface of the inertia ellipsoid. The modulus of the arguments u and s are k and k' given by

$$k^2 = \frac{(A-B)C}{(A-C)B}, \quad k'^2 = 1 - k^2 \tag{11}$$

Introducing the new variables x, y such that

$$x = \int_0^u \sqrt{B - (A-B)sn^2u}\, du,$$

$$y = \int_0^s \sqrt{A - Bk'^2sn^2s}\, \frac{ds}{dn\ s}$$

with artificial time τ related with the real time t by the relation

$$d\tau = \frac{dt}{I}, \tag{12}$$

where

$$I = \frac{Ck}{\Lambda}\left(k'^2 \frac{sn^2s}{dn^2s} + cn^2u\right), \tag{13}$$

and

$$\Lambda = \frac{1}{dn^2s}(Ak^2sn^2u + Bk^2cn^2usn^2s + C\ dn^2ucn^2s). \tag{14}$$

Under the above relations, the equations of motion (8) take the following formula:

$$x'' - Fy' = V_x, \qquad y'' + Fx' = V_y \tag{15}$$

where the prime denotes differentiation with respect to τ. Equations (15) describe the motion of fictitious material point in a plane under the action of conservative force V and gyroscopic force F, where

$$V = \frac{1}{k}\left[h - \frac{mg}{dn^2s}(ak^2sn^2u + bk^2sn^2s\ cn^2u + c\ dn^2ucn^2s) - \frac{n^2}{2\Lambda}\right], \tag{16}$$

$$F = \frac{n}{\mu_1(u)\mu_2(s)}\left\{v_2(s)\frac{\partial}{\partial u}\left[\frac{\lambda_1(u)}{\Lambda M}\right] + v_1(u)\frac{\partial}{\partial s}\left[\frac{\lambda_2(s)}{\Lambda M}\right]\right\}, \tag{17}$$

$$v_1(u) = Cdn^2u + k^2(A-B)sn^2ucn^2u,$$

$$v_2(u) = Ccn^2s - k^2(A-b)\frac{sn^2s}{dn^2s},$$

and where

$$\lambda_1(u) = sn^2ucn^2u\ dn^2u, \qquad \lambda_2(s) = \frac{sn^2scn^2s}{dn^2s}, \qquad M = 1 - cn^2ucn^2s,$$

$$\mu_1^2(u) = B - (A-B)sn^2u, \qquad \mu_2^2(s) = \frac{1}{dn^2s}(A - Bk'^2sn^2s).$$

The transformed system (15) admits the Jacobi integral

$$x'^2 + y'^2 = 2V.$$

2. PERIODIC SOLUTIONS

To get the periodic solutions in the vicinity of equilibrium points of System (15), we shall use Liapunov's (1966) method of holomorphic integral. This method has been successfully used in the problem of celestial mechanics and recently in mechanics of a rigid body (El-Sabaa 1989).

Equilibrium positions of System (15) are given by

$$V_x = 0, \qquad V_y = 0, \qquad V = 0 \tag{18}$$

which can be written in the form

$$W_x = 0, \qquad W_y = 0, \qquad W + h = 0 \tag{19}$$

where

$$W = -\frac{mg}{dn^2 s}(ak^2 sn^2 u + bk^2 sn^2 scn^2 u + c\, dn^2 ucn^2 s) - \frac{n^2}{2\Lambda}\cdot \tag{20}$$

The first two equations of (19) determine the equilibrium points P_r, where the coordinates of these points are given by $(x_{0r}(n), y_{0r}(n))$, for a certain value of n. Equation $W + h = 0$ gives the value of h for each of the points P_r.

To get the periodic solution with a fixed value of parameter n, in the vicinity of the point P_0, we put

$$x = x_0 + \xi, \qquad y = y_0 + \eta$$

in System (15), then we have

$$\begin{aligned} \xi'' - F(x_0 + \xi, y_0 + \eta) &= V_\xi(x_0 + \xi, y_0 + \eta; h) \\ \eta'' + F(x_0 + \xi, y_0 + \eta) &= V_\eta(x_0 + \xi, y_0 + \eta; h) \end{aligned} \tag{21}$$

where the first integral of System (15) becomes

$$\xi'^2 + \eta'^2 - 2V(x_0 + \xi_1 y_0 + \eta; h) = 0 \tag{22}$$

By using Liapunov's (1966) method on holomorphic integrals, we get the periodic solution of System (21). The seeking solution is a power series of parameter μ

$$\xi = \sum_{s=1}^{\infty} \mu^s x^{(s)}, \qquad \eta = \sum_{s=1}^{\infty} \mu^s y^{(s)} \tag{23}$$

and

$$h = h_0 + \sum_{s=2}^{\infty} \mu^s h_s$$

where h_s is a constant, and h is the Jacobi's constant for the perturbed motion. The functions $x^{(s)}$, $y^{(s)}$ are periodic functions in τ with period T given by the formula

$$T = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0}\left(1 + \sum_{s=2}^{\infty} T_s \mu^s\right) \tag{24}$$

Introducing the new argument $v = \lambda\tau$, then the seeking solution can be expressed in terms of Fourier series

$$x^{(s)} = a_{1s}^0 + \sum_{r=1}^{\infty} (a_{1s}^{(r)} \cos rv + b_{1s}^{(r)} \sin rv),$$

$$y^{(s)} = a_{2s}^0 + \sum_{r+1}^{\infty} (a_{2s}^{(r)} \cos rv + b_{2s}^{(r)} \sin rv). \tag{25}$$

A first approximation to the system will be

$$\lambda_0^2 \frac{d^2x^{(1)}}{dv^2} - F_0 \lambda_0 \frac{dy^{(1)}}{dv} + lx^{(1)} + my^{(1)} = 0$$

$$\lambda_0^2 \frac{d^2y^{(1)}}{dv^2} + F_0 \lambda_0 \frac{dx^{(1)}}{dv} + mx^{(1)} + py^{(1)} = 0 \tag{26}$$

where the quantities l, m, and p are given by

$$l = -I_0\left(\frac{\partial^2 W}{\partial x^2}\right)_{\substack{x=x_0,\\ y=y_0}}, \qquad m = -I_0\left(\frac{\partial^2 W}{\partial x \partial y}\right)_{\substack{x=x_0,\\ y=y_0}}, \qquad p = -I_0\left(\frac{\partial^2 W}{\partial y^2}\right)_{\substack{x=x_0,\\ y=y_0}}, \tag{27}$$

and I_0 is the value of I at $x = x_0$, $y = y_0$.

The characteristic equation of (26) is given by

$$\lambda_0^4 + (l + p + F_0^2)\lambda_0^2 - (lp - m^2) = 0. \tag{28}$$

Liapunov's (1966) theorem of holomorphic integral states that, if the force function for a system has a maximum value at equilibrium points (i.e. the equilibrium points are stable), then there are two different frequencies, where each frequency gives a family of periodic solutions. The frequencies of the system are given by

$$\lambda_0 = \sqrt{\frac{l + p + F_0^2 \pm \sqrt{(l + p + F_0^2)^2 - 4(lp - m^2)}}{2}} \tag{29}$$

If we consider the case

$$lp - m^2 > 0 \tag{30}$$

with the condition

$$F_0^2 > 2\sqrt{lp - m^2} - l - p, \tag{31}$$

then we obtain from (30) and using (27) the following inequality:

$$(W_{xx} W_{yy})_{\substack{x=x_0\\ y=y_0}} > [(W_{xy})_{\substack{x=x_0\\ y=y_0}}]^2. \tag{32}$$

This means W has an extremum at P_0. In addition, (31) implies $(W_{xx})_{x=x_0,\, y=y_0} < 0$, which yields W has a maximum value at P_0. The inequality (31) then gives two different frequencies.

In the case of $lp - m^2 < 0$, we have a single frequency stemming from (29) with plus sign, where in this case, the function W has a form of a saddle at P_0. In the first case, we have two different frequencies given by (29), where every frequency λ_0 corresponds to a family of periodic solutions, while in the second case we have only one family.

The periodic solutions in the first approximation are written as follows:

$$x = x_0 + \mu(\lambda_0^2 - p)\sin v$$
$$y = x_0 + \mu(m \sin v - F_0 \lambda_0 \cos v). \tag{33}$$

On the inertia ellipsoid, the above solutions represent similar ellipses with a common centre P_0.

3. CONCLUSION

Although the problem of rotational motion of a rigid body has been solved in the three mentioned cases, the geometric interpretation of the solution still is not clear. The modern methods of qualitative analysis of nonlinear systems of differential equations have their origins in the work of Poincaré (1892–1899), Birkhoff (1927), Liapunov (1966), Andronov & Pontryagin (1937) and Arnold (1963). In many problems of a rigid body, we consider the periodic solutions and their stability as fundamental structures of the system, where the most important information about the motion is attained, i.e. orbits around stable periodic orbits have similar shape, while orbits starting close to unstable periodic orbits belong to the general sea of chaotic orbits.

To construct the periodic solutions in the neighbourhood of the equilibrium points, the Liapunov's method is an appropriate one, especially because the stability of the periodic motion can be obtained directly from the characteristic equation (18). Many problems of celestial mechanics and rigid bodies were solved by Poincaré method of small parameter, where the periodic solutions were constructed by Demin & Kiselev (1974) and by Barkin & Ievlev (1977) who employed the variables of Andoyer (1923) and Deprit (1967). In the last ten years, the numerical computation to integrate the equations of motion was quickly paired with Poincaré's (1892–1899) surface-of-section method, or the fixed point method (Hénon & Heiles 1964). The periodic orbits and their stability can be obtained numerically, because in this method we do not consider the whole trajectory in phase space, but only its consecutive crossings of a definite surface, in particular, with a plane. Equation (15) can be solved numerically by using the above method to obtain its periodic orbits and their stability will be treated in a forthcoming paper.

REFERENCES

Andoyer, M. 1923. Cours de mecanique celeste, vol. 1. Gauthiers-Villars, Paris.

Andronov, A. & Pontryagin, L. 1937. Systemes Grossiers. Dikladu Akademia Nauk SSR **14**: 247–51.

Arnold, V. 1963. Small divisor problems in classical and celestial mechanics. Russian Mathematical Survey **18**: 581–85.

Barkin, Iu. & Ievlev, V. 1977. Periodic motions of a rigid body with a fixed point in the gravity field of two centers. Prikladnia Matematik e Mekhanik **41**: 558–61.

Birkhoff, G. 1927. Dynamical systems. American Mathematical Society Publications, Providence, USA.

Demin, V. & Kiselev, F. 1974. On periodic motions of a rigid body in central Newtonian field. Prikladnia Matematik e Mekhanik **38**: 224–27.

Deprit, A. 1967. Free rotation of a rigid body studied in the phase plane. American Journal of Physics **35**: 424–28.

El-Sabaa, F. 1989. About the periodic solutions of Kowalevskaya's top by using Liapunov's method. Journal of the University of Kuwait (Science) **16**: 21–27.

Golubev, V. 1953. Lectures on integration of the equations of motion of a rigid body about a fixed point. Mir Publications, Moscow.

Hénon, M. & Heiles, C. 1964. An applicability of the third integral of motion: some numerical experiments. The Astronomical Journal **29**: 73–79.

Khalimanovich, M. 1953. On the motion of a non-completely symmetric heavy gyroscope at small angles of nutation. Journal of Belorusskee University **15**: 10–15.

Kharlamov, P. 1963. On the equations of motion for a heavy body with fixed point. Prikladnia Matematik e Mekhanik **27**: 1070–78.

Kolossoff, G. 1903. On some modification of Hamilton's principles applied to the problem of mechanics of a rigid body. Troda Otdelenii Fiziecheskikh Nauk Obshechestva Liubitelei Estesvoznaniia **1**: 5–12.

Liapunov, A. 1966. Stability of motion. Academic Press, New York.

MacMillan, W. 1936. Dynamics of rigid bodies. McGraw-Hill, New York.

Poincaré, H. 1892–1899. Les méthodes nouvelles de la mécanique céleste. Gauthier-Villars, Paris.

Toporova, V. 1962. On a new case of exact integrability of rotation of a heavy rigid body about a fixed point. Trudy Institut Matematik Akademia Nauk UZSSR **24**: 15–25.

Vagner, E. & Demin, V. 1975. On a class of periodic motions of a solid body about a fixed point. Prikladnia Matematik e Mekhanik **39**: 890–93.

Whittaker, E. 1965. A treatise on the analytical dynamics of particles and rigid bodies. Cambridge University Press, London.

(Received 7 September 1988, revised February 1989)

عن الحلول الدورية لدوران الجسم الثقيل المتماسك حول نقطة ثابتة

فوزي محمد فهمي السبع
قسم الرياضيات بجامعة الكويت
ص . ب . ٥٩٦٩ ، الصفاة ١٣٠٦٠ ، الكويت

خلاصة

حولت معادلات الحركة للجسم الثقيل المتماسك والذي يدور حول نقطة ثابتة الى معادلات الحركة لنقطة مادية في مستوى ، وذلك باستخدام الاحداثيات الايسوثرمالية على سطح ناقص القصور الذاتي . وباستخدام طريقة ليبونوف للتكاملات الهولومـورفية ، أمكن ايجـاد عائلة من الحلول الدورية .