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A comparison of strength of material, elasticity and finite element analysis

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ABSTRACT

The magnitude of stress distribution as predicted by the simplified strength of material theory was compared with that predicted by the theory of elasticity and was checked by a proposed set of test problems using the finite element method. Stress analysis was applied in this study to several practical problems, and a graphical comparison was made between stresses using the three methods of analysis. Results showed that the magnitude of stresses depends on the choice of the theory used, and the difference in stresses using the three methods of analysis reflects the degree of approximation involved in each method. For those problems where planes before loading remain plane after loading, the simplified strength of material theory provides adequate solutions.

INTRODUCTION

The stress distribution in linear elastic bodies is commonly determined by one of there procedures: the use of strength of material theory which is sometimes referred to as mechanics of material theory, the continuum mechanics approach which is referred to as the theory of elasticity, and the approximate numerical technique, such as the finite element or the finite difference method. All these procedures have to satisfy: (a) Equilibrium conditions, and (b) Compatibility conditions.

In the strength of materials procedure condition (a) is insured by satisfying the equations of equilibrium for the whole member, while condition (b) is satisfied by a simple basic assumption related to the geometry of the deformed member. This assumption states that a cross sectional plane before loading remains plane after loading (Timoshenko & Goodier 1970). Thus, while the strength of materials procedure is accurate for the case of single loading of a straight member with uniform cross section, it is approximate for cases when warpage occurs in cross sections during loading.

In the theory of elasticity procedure condition (a) is insured by satisfying the differential equations of equilibrium for small volumetric element, while condition (b) is satisfied by the differential equation of compatibility of an elastic body (Popov

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* Deceased.

1978). The theory of elasticity considers strains from the mathematical point of view which is based on the exact kinematics of small deformation and the conservation of linear and angular momentum. The differential equations of equilibrium and compatibility conditions must be solved for the specific boundary conditions of the problem at hand. The theory of elasticity approach involves fewer assumptions than the strength of materials approach, which makes it more accurate; however, the mathematical solution is so complicated that closed form solutions are available only for a few problems with simple displacement and force boundary conditions.

The basic concept of the finite element method, which is selected for the numerical procedure in this study is that the continuum (total structure) be subdivided into small regions (finite elements). The behaviour of each element is described by functions representing the displacement or stresses in the region (Cook 1981). In the displacement formulation of the finite element, it is generally considered that the displacement and not the stresses are continuous. Unlike the theory of elasticity, the finite element procedure is not restricted by the shape or boundary conditions of the problem (Zienkiewicz 1977).

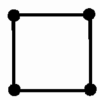
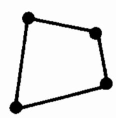
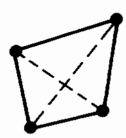
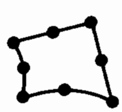
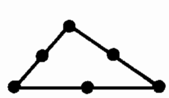
Because of the simplicity and ease of application of the strength of materials theory, it is used more widely in conventional design than the theory of elasticity, even though the theory of elasticity provides more accurate results. However, for curved members, or members with variable cross section, the strength of materials theory cannot provide reliable results. In most practical problems the boundary conditions are too complicated to be incorporated in closed form mathematical solutions; thus numerical procedure becomes necessary. Although, the numerical solution is tedious, the availability of more powerful and faster computers makes the finite element method most appropriate for the solution of engineering problems. It is important to note, however, that the success of the finite element analysis depends on the selection of the elements and the discretizing assumptions used. It is also essential that the accuracy of the finite element analysis be verified by an independent and well established analytical procedure.

INFLUENCE OF ELEMENT PROPERTIES

Commercially developed finite element programs have become widely available in the engineering community. These programs were found (Conner & Will 1969; Robinson & Blackham 1979, 1981) to provide stress analysis results ranging from excellent to poor. Thus in order to ascertain the accuracy of a particular finite element program in various applications, a standard set of finite element problems has to be designed and tested by the user prior to using the program. One popular commercially available finite element program, known as STRUDL (Logcher *et al.* 1979), has been used in this study.

The design of an adequate set of problems for finite element testing requires careful planning; the initial testing is usually done with one or two relatively easy problems with known solutions. For example, problems used in testing the STRUDL bending problems were single type elements, even though the accuracy of the finite element depends largely on the element numbers, shapes and grid spacing. Hence, in order to determine the effect of element shapes, several STRUDL ele-

Table 1. Properties of element used in the finite element procedure

Element shape	Element name	No. of nodes	Element description
	PSR	4	Rectangular element whose stiffness matrix is based on a displacement expansion, which produces quadratic displacement field over the element and a linear displacement variation along the edges.
	IPLQ	4	Isoparametric with expansion that produces a linear displacement field over the element and a linear displacement variation along the edges.
	PSQ1	4	Composed of two pairs of overlapping triangles, with expansion that produces a displacement expansion function with a linear displacement field.
	IPQQ	8	Isoparametric with expansion displacement field over the element and quadratic displacement variation along the edges.
	CSTG	3	Triangular element with displacement expansion function which produces a linear displacement field.
	LST	6	Isoparametric with displacement expansion function which produces a quadratic displacement field.

ments, shown in Table 1, were selected and used in this work. The influence of elements characteristics on the stress analysis is not known and will be one of the objectives of this study. Elements selected include rectangular element with four nodal points (PSR), quadrilateral-isoparametric element with four nodal points (IPLQ), quadrilateral element generated from the overlaid combination of four triangular elements (PSQ1), quadrilateral element with four corner nodal points and four mid-span nodal points (IPQQ), triangular element with three nodal points (CSTG), and triangular element with three corner nodal points and three mid-side nodal points (LST). Properties of each of the six elements are summarized in Table 1.

PURPOSE OF STUDY

The major objective of this study is to compare the magnitude and the distribution of stresses within bodies of different shapes and boundary conditions as predicted by the strength of materials theory, as well as the theory of elasticity and the finite element method. Two-dimensional plane stress problems selected for the study have well known analytic solutions which allow comparison with the numerical solution.

Another purpose of the study, is to determine the influence of the type of element used on the accuracy of the finite element results. It is hoped that the information presented will provide a better understanding of how these three methods of stress analysis are related and which approach should be selected in the design of practical problems.

TWO-DIMENSIONAL PROBLEMS

Two-dimensional plane stress problems selected for comparison are the bending of straight beams and bending of curved beams having narrow cross sections. The straight beams selected include a cantilever beam with concentrated force at the end, a simply supported beam with uniform load, a cantilever beam with linearly increasing load, and a tapered cantilever beam with uniform load. Curved beams utilized for the comparison are curved beam subjected to pure bending and curved cantilever beam subjected to a concentrated force. In order to provide better comparison all graphs generated in the study are presented in dimensionless form.

CANTILEVER BEAM WITH CONCENTRATED FORCE AT THE END

Consider the cantilever beam with length L and depth $2c$, subjected to concentrated force P as shown in Fig. 1. The normal stress σ_x and shear stress τ_{xy} acting on the x-plane, using the strength of material theory, are given by the following equations:

$$\sigma_x = -\frac{My}{I} \tag{1a}$$

$$\sigma_y = 0 \tag{1b}$$

$$\tau_{xy} = \frac{VQ}{Ib} \tag{1c}$$

where M and V are the bending moment and the shear force respectively, acting on a section of the beam, I is the moment of inertia about the neutral axis of the section,

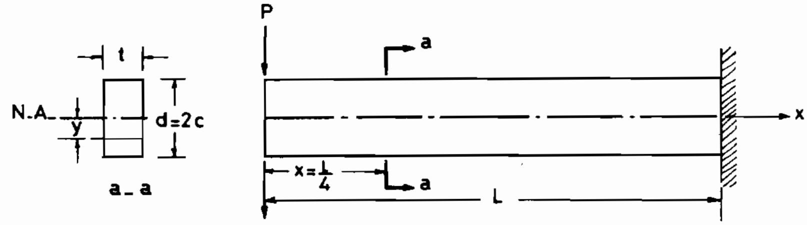


Fig. 1. Cantilever beam with concentrated force at the end.

Q is the first statical moment about the neutral axis of an area extended from the top of the beam to distance y from the neutral axis (Fig. 1), $2c$ is the depth of the beam and t is the thickness of the cross section which is assumed to be unity in this analysis.

Applying Eqn 1 to the beam shown in Fig. 1, the normal and shear stresses according to the strength of materials theory (Timoshenko & Goodier 1970) may be expressed as follows:

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3}xy \tag{2a}$$

$$\sigma_y = 0 \tag{2b}$$

$$\tau_{xy} = -\frac{3}{4}\frac{P}{c^3}(c^2 - y^2) \tag{2c}$$

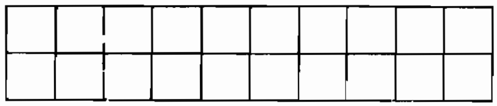
From the theory of elasticity (Popov 1978) the normal and shear stresses

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3}xy \tag{3a}$$

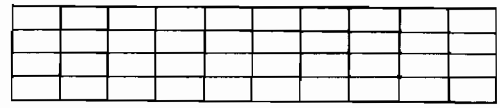
$$\sigma_y = 0 \tag{3b}$$

$$\tau_{xy} = -\frac{3}{4}\frac{P}{c^3}(c^2 - y^2) \tag{3c}$$

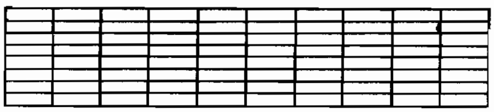
It is clear from Eqns 2 and 3 that the normal and shear stresses predicted by both theories are identical. Thus it was decided to select the case of the cantilever beam subjected to a force at the end to become the first standard test problem in this work. This choice is intended to serve two principal functions. In the first place, it is essential to assess the accuracy of the choice of the element used in the solution; also the selection of the element pattern and model used to represent the continuum. It is obvious that the accuracy will improve as the number of the element is increased. However, it must be recognized that the location, orientation, and the aspect ratio of an element are also important factors in the finite element solution. Therefore, in order to develop a technique for selecting an appropriate element discretization which is accurate as well as economical with respect to the execution time it was thought that the problem selected must have an identical solution using both the strength of materials theory and the theory of elasticity, thus enabling easy assessment of the finite element solution. The element patterns, numbers and the models representing the first test problem are shown in Fig. 2, while the aspect ratio of the selected models and the different types of elements are given in Table 2. Using these various models (Fig. 2 and Table 2) in the STRUDL program, the variation of normal and shear stress along the cross section of the beam obtained is presented in Fig. 3. On the same figure, results obtained from the strength of materials theory and the theory of elasticity are also shown. The ratios of the normal stress σ_x computed by the finite element method to that computed by the theory of elasticity along a section at distance $L/4$ from the origin is shown in Table 2. This table shows that selecting several types of STRUDL elements will not change the results significantly and that the accuracy of the solution for element PSQ1, PSR, IPLQ and IPQQ depends on the aspect ratio and the number of elements. This is confirmed in



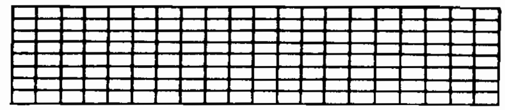
Case 1. 33 NODES, 20 ELEMENTS.



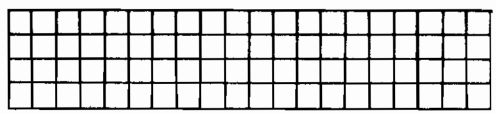
Case 2. 55 NODES, 40 ELEMENTS.



Case 3. 99 NODES, 80 ELEMENTS.



Case 4. 189 NODES, 160 ELEMENTS.



Case 5. 105 NODES, 80 ELEMENTS.

Fig. 2. Typical element numbers and meshers.

Fig. 3 which shows that using cases 4 and 5 (Fig. 2), in the STRUDL solution yields results which compared very well with the exact solution. It is also clear from the same figure that the results obtained using the STRUDL solution from case 2 are reasonably accurate, while those obtained from case 1 deviated slightly from the exact solution.

SIMPLY-SUPPORTED BEAM WITH UNIFORM LOAD

Consider a simply-supported beam of length $2L$ having a narrow rectangular cross section as shown in Fig. 4 with thickness t and depth $2c$. The normal and shear stresses acting on the x-plane can be obtained using the strength of materials theory as follows:

$$\sigma_x = -\frac{q}{2I}(L^2 - x^2)y = -\frac{3q}{4c^3}(L^2 - x^2) \tag{4a}$$

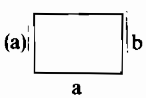
$$\sigma_y = 0 \tag{4b}$$

$$\tau_{xy} = -\frac{qx}{2I}(c^2 - y^2) = -\frac{3qx}{4c^3}(c^2 - y^2) \tag{4c}$$

If a convenient stress function is chosen, the theory of elasticity (Popov 1978)

Table 2. Ratios of the normal stress σ_x at $L/4$ from the origin of the cantilever beam as computed by finite element method to those computed by elasticity

Case	Aspect ratio[a] (a/b)	Type of element	No. of elements	No. of nodes	Stress σ_x at $L/4$ (F.E./Elast.)
1	1	PSQ1	20	33	0·901
		PSR		33	0·912
		IPLQ		33	0·913
		IPQQ		85	0·913
2	2	PSQ1	40	55	0·908
		PSR		55	0·930
		IPLQ		55	0·930
		IPQQ		149	0·930
3	4	PSQ1	80	99	0·921
		PSR		99	0·934
		IPLQ		99	0·934
		IPQQ		277	0·934
4	2	PSQ1	160	189	0·979
		PSR		189	0·996
		IPLQ		189	0·996
		IPQQ		537	0·996
5	1	PSQ1	80	105	0·987
		PSR		105	0·999
		IPLQ		105	0·999
		IPQQ		289	0·999



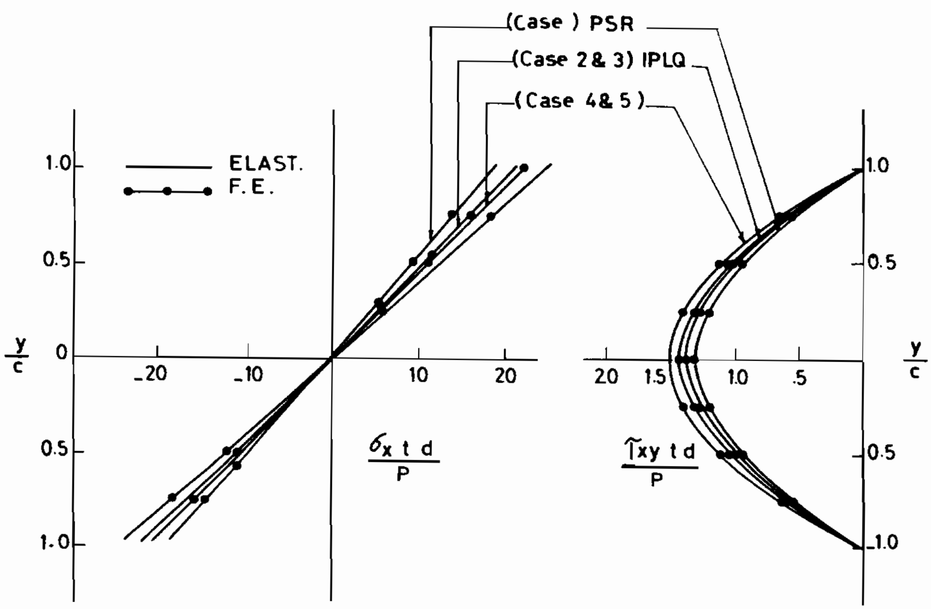


Fig. 3. Variation of σ_x and τ_{xy} for cantilever beam.

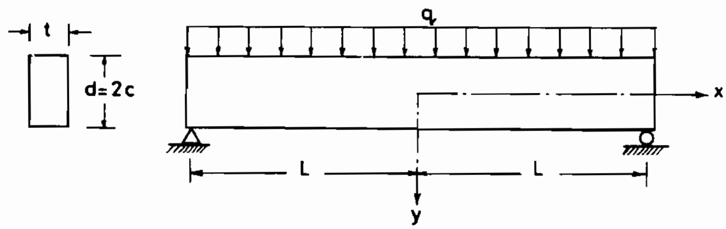


Fig. 4. Simply-supported beam with uniformly distributed load.

provides values for the normal and shear stresses as follows:

$$\sigma_x = \frac{q}{2I}(L^2 - x^2)y + \frac{q}{I}\left(\tfrac{1}{3}y^3 - \tfrac{1}{5}c^2y\right) \tag{5a}$$

$$\sigma_y = -\frac{q}{2I}\left(\tfrac{1}{3}y^3 - c^2y + \tfrac{2}{3}c^3\right) \tag{5b}$$

$$\tau_{xy} = -\frac{qx}{2I}(c^2 - y^2) \tag{5c}$$

It is clear from Eqns 4 and 5 that the shear stresses predicted by the two theories are identical, whereas the normal stresses are different. This is the reason why the case of the simply-supported beam with uniform load was selected as a second test problem in this study. The models representing the second test problem are shown in Fig. 2. The aspect ratio of the selected models, the different types of elements used in the finite element procedure and the ratio of σ_x calculated by the finite element to that calculated by the theory of elasticity are given in Table 3.

Case 5 with PSR and IPLQ elements was used for comparison with the theory of elasticity and strength of material (Fig. 5).

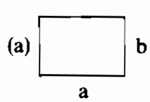
The variation of the normal stress σ_x and σ_y, along the cross section of the beam as obtained by the STRUDL solution, the strength of materials theory and the theory of elasticity, are shown in Fig. 5 and Fig. 6 respectively. It is clear from Fig. 5 that the normal stresses σ_x obtained using the strength of materials theory are exactly the same as those obtained from the theory of elasticity. Using case 5 the normal stresses σ_x and σ_y for several beams with length to depth ratio of 2, 5, 10 are investigated and the results of the analyses are shown in Figs 7 and 8. These figures indicate that there is slight difference between results of the finite element method and those obtained from the theory of elasticity.

CANTILEVER BEAM WITH LINEARLY INCREASING LOAD

Consider a cantilever beam with length l, depth $2c$, and thickness t, subjected to a linearly increasing load of intensity γl as shown in Fig. 9.

Table 3. Ratios of the normal stress σ_x at $L/2$ of the simply-supported beam as computed by the finite element method to that computed by the theory of elasticity

Case	Aspect ratio[a] (a/b)	Type of element	No. of elements	No. of nodes	Stress σ_x at $L/2$ (F.E./Elast.)
1	1	PSQ1	20	33	0·900
		PSR		33	0·914
		IPLQ		33	0·914
		IPQQ		85	0·920
2	2	PSQ1	40	55	0·923
		PSR		55	0·938
		IPLQ		55	0·941
		IPQQ		149	0·947
3	4	PSQ1	80	99	0·935
		PSR		99	0·950
		IPLQ		99	0·950
		IPQQ		277	0·956
4	2	PSQ1	160	189	1·031
		PSR		189	1·047
		IPLQ		189	1·047
		IPQQ		537	1·054
5	1	PSQ1	80	105	1·016
		PSR		105	1·032
		IPLQ		105	1·032
		IPQQ		289	1·039



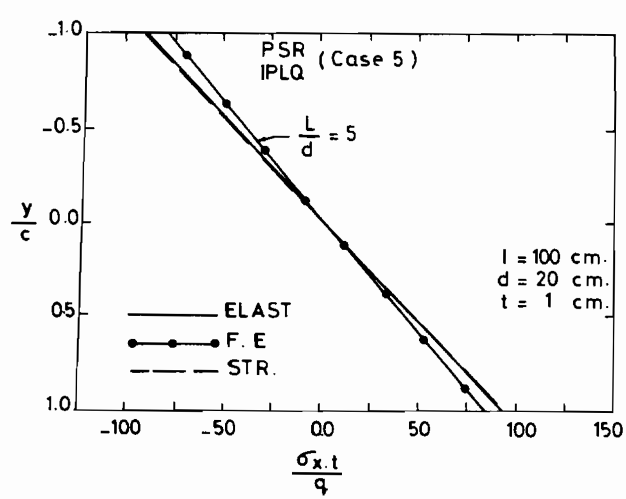


Fig. 5. Variation of σ_x for simply-supported beam.

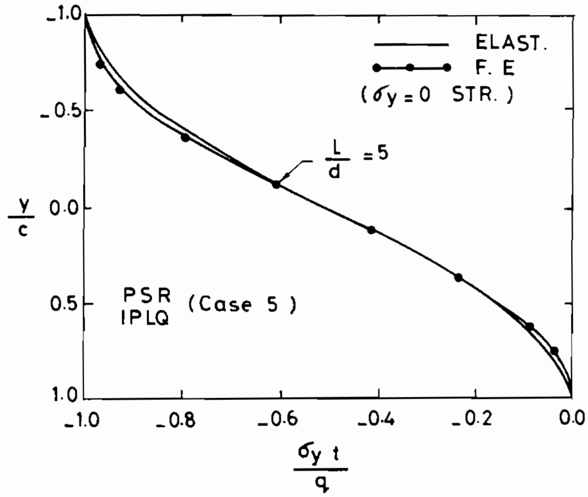


Fig. 6. Variation of σ_y for simply-supported beam.

The normal and shear stresses acting on the x-plane, using the strength of materials theory, can be determined from Eqn. 1 as follows:

$$\sigma_x = \frac{\gamma x^3 y}{6I} = \frac{\gamma x^3 y}{4c^3} \tag{6a}$$

$$\sigma_y = 0 \tag{6b}$$

$$\tau_{xy} = \frac{\gamma x^2}{4I}(c^2 - y^2) = \frac{3\gamma x^2}{8c^3}(c^2 - y^2) \tag{6c}$$

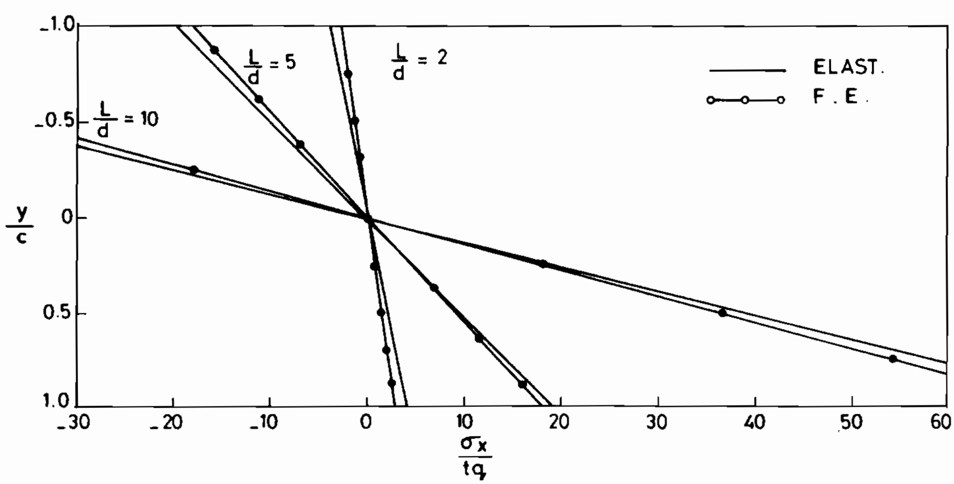


Fig. 7. Variation of σ_x for simply-supported beam with different length to width ratio.

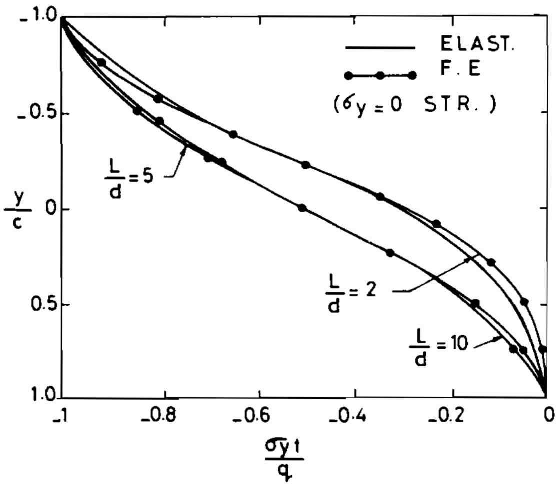


Fig. 8. Variation of σ_y for simply-supported beam with different length to width ratio.

Applying the theory of elasticity (Popov 1978) for the beam, shown in Fig. 9, the following expression for the normal and shear stresses are obtained:

$$\sigma_x = \frac{\gamma x^3 y}{4c^3} + \frac{\gamma}{4c^3}\left(\tfrac{6}{5}c^2xy - 2xy^3\right) \tag{7a}$$

$$\sigma_y = \gamma x\left(\frac{y^3}{4c^3} - \frac{3y}{4c}\right) - \frac{\gamma x}{2} \tag{7b}$$

$$\tau_{xy} = \frac{3\gamma x^2}{8c^3}(c^2 - y^2) - \frac{\gamma}{8c^3}(c^4 - y^4) + \frac{3\gamma}{20c}(c^2 - y^2) \tag{7c}$$

It is evident from Eqns 6 and 7 that the expressions for the normal stress σ_x, σ_y and shear stress τ_{xy} as computed by the strength of materials theory and the theory of elasticity are different. For this reason, it was decided to select the case of the cantilever beam with linearly increasing load to become the third standard test problem in this work. The models shown in Fig. 2, and the aspect ratio of elements shown in Table 4, were used in the finite element analysis of this problem.

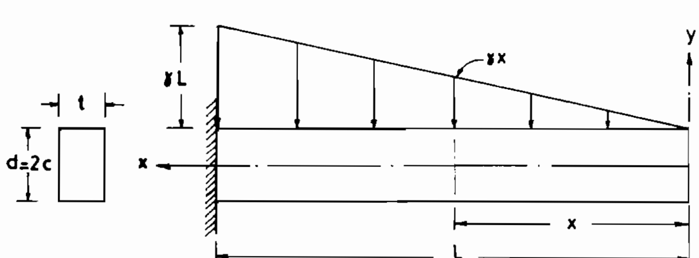
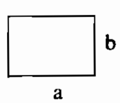


Fig. 9. Cantilever beam with linearly-increasing load.

Table 4. Ratios of the normal stress σ_x at distance $L/2$ from the origin of a cantilever beam subjected to linearly increasing load and as computed by the finite element method to those computed by elasticity

Case	Aspect ratio[a] (a/b)	Type of element	No. of elements	No. of nodes	Stress σ_x at $L/2$ (F.E./Elast.)
1	1	PSQ1	20	33	0·867
		PSR		33	0·880
		IPLQ		33	0·880
		IPQQ		85	
2	2	PSQ1	40	55	0·903
		PSR		55	0·917
		IPLO		55	0·917
		IPQQ		149	0·923
3	4	PSQ1	80	99	0·911
		PSR		99	0·925
		IPLQ		99	0·925
		IPQQ		277	0·931
4	2	PSQ1	160	189	0·952
		PSR		189	0·967
		IPLQ		189	0·967
		IPQQ		537	0·973
5	1	PSQ1	80	105	0·955
		PSR		105	0·970
		IPLQ		105	0·970
		IPQQ		289	0·976

(a)


The variation of σ_x, σ_y and τ_{xy} using the three types of analysis are presented in Figs 10, 11 and 12 respectively. Table 4 reflects the effect of increasing the number of elements on the accuracy of the solution. The results indicate that ordinary rectangular elements with four nodal points will result in sufficiently accurate solution and using quadrilateral-isoparametric elements is not justified in such problems. Figs 10, 11 and 12 show that using case 5 in the STRUDL solution renders sufficiently accurate results if compared with the theory of elasticity solution. It is also shown in Fig. 10, that the strength of materials, and the theory of elasticity provide approximately similar values of normal stress σ_x while the values of shear stress τ_{xy} (Fig. 12) differ significantly.

TAPERED CANTILEVER BEAM WITH UNIFORM LOAD

Consider a tapered cantilever beam loaded with a uniform load q and wedge angle β as shown in Fig. 13. The normal and shear stresses at any radial distance θ, as calculated by the strength of materials theory can be expressed as

$$\sigma_x = \frac{6q}{t} \cot^3\beta \left(\tfrac{1}{2} \tan \beta - \tan \theta\right) \tag{8a}$$

$$\sigma_y = 0 \tag{8b}$$

$$\tau_{xy} = -\frac{6q}{t} \cot^3\beta \left(\tan \beta \tan \theta - \tan^2 \theta\right) \tag{8c}$$

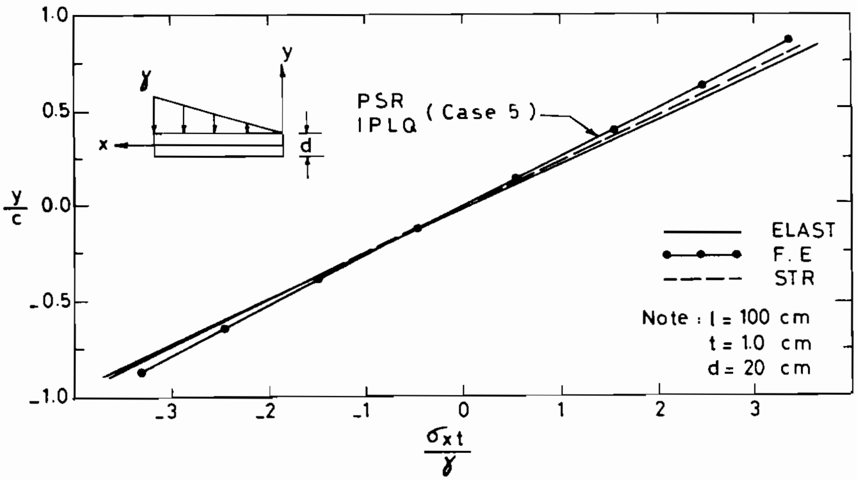


Fig. 10. Variation of σ_x for cantilever beam with linearly-increasing load.

From the theory of elasticity the normal and shear stresses can be expressed as

$$\sigma_x = \frac{q}{t(\beta - \tan\beta)}\left[\tfrac{1}{2}\sin 2\theta + (\theta - \beta)\right] \tag{9a}$$

$$\sigma_y = \frac{q}{t(\beta - \tan\beta)}\left[\tan\beta - \tfrac{1}{2}\sin 2\theta + (\theta - \beta)\right] \tag{9b}$$

$$\tau_{xy} = \frac{q}{t(\beta - \tan\beta)}\sin^2\theta \tag{9c}$$

The mesh size and the number of elements used in the STRUDL program for the analysis of the problem are shown in Fig. 14, and the different types of elements used for the numerical solution are given in Table 5.

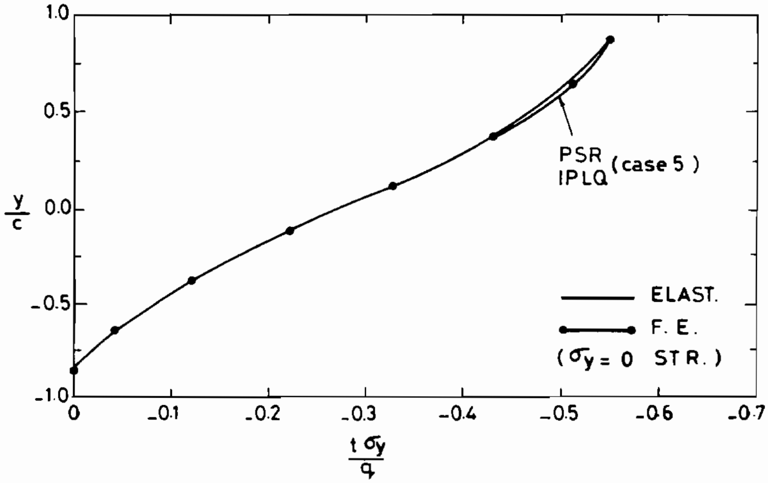


Fig. 11. Variation of σ_x for cantilever beam with linearly-increasing load.

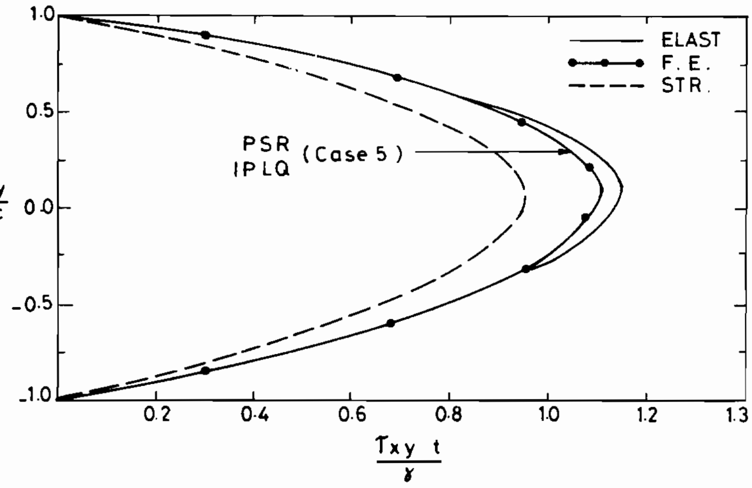


Fig. 12. Variation of τ_{xy} for cantilever beam with linearly – increasing load.

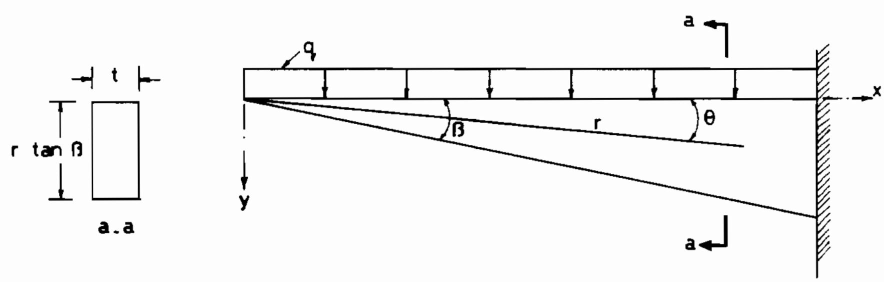
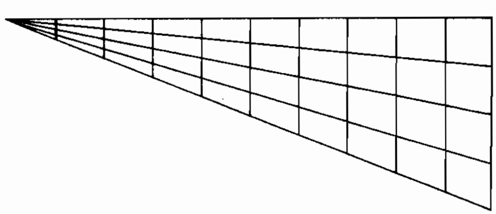


Fig. 13. Tapered cantilever beam with uniform load.



Case 1. 51 NODES, 40 ELEMENTS.
Case 2. 91 NODES, 80 ELEMENTS.
Case 3. 141 NODES, 40 ELEMENTS.
Case 4. 281 NODES, 80 ELEMENTS.

Fig. 14. Typical element numbers and meshes for tapered beam.

Table 5. Ratios of the normal stress σ_x at distance $L/2$ from the origin and $\beta/2$ for a tapered cantilever beam subjected to uniform load and as computed by the finite element method to those computed by elasticity

Case	Type of element	No. of elements	No. of nodes	Stress σ_x at $L/2$ and $\beta/2$ (F.E./Elast.)
1	CSTG	4	51	2·14
	PSR	36		
2	CSTG	4	51	0·91
	PSQ1	36		
3	CSTG	4	51	0·98
	IPLQ	36		
4	CSTG	4	91	2·03
	PSR	76		
5	CSTG	4	91	0·92
	PSQ1	76		
6	CSTG	4	91	0·99
	IPLQ	76		
7	LST	6	141	0·93
	IPQQ	36		
8	LST	6	281	1·03
	IPQQ	76		

Table 5 shows the ratios of the normal stress σ_x at a distance $x = L/2$ and angle $\beta/2$ (Fig. 15), where $L = 100$ cm and $\beta = 30$ or $60°$. The results indicate that it is more appropriate to use the quadrilateral-isoparametric element for the analysis of the tapered cantilever beam. The results obtained using standard square element PSR deviate significantly from the theoretical results regardless of the number of

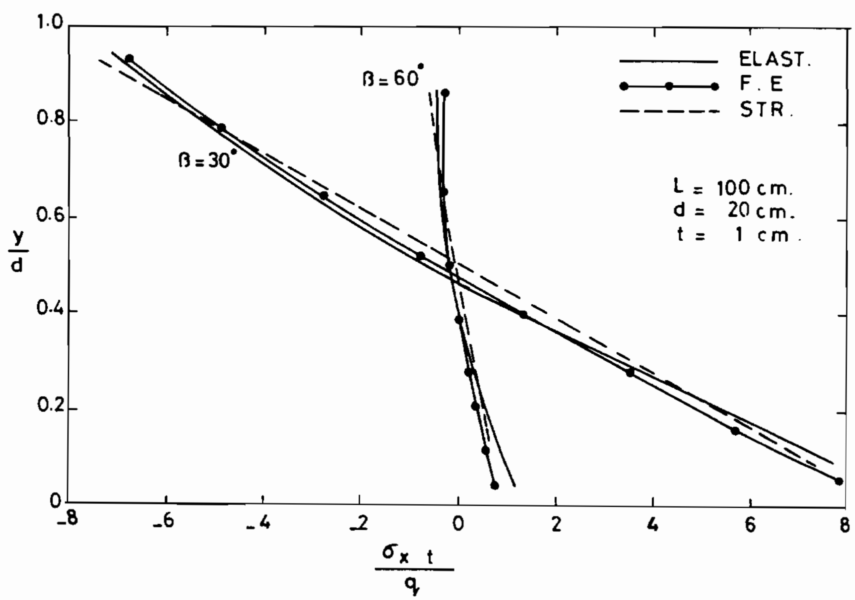


Fig. 15. Variation of σ_x for tapered beam.

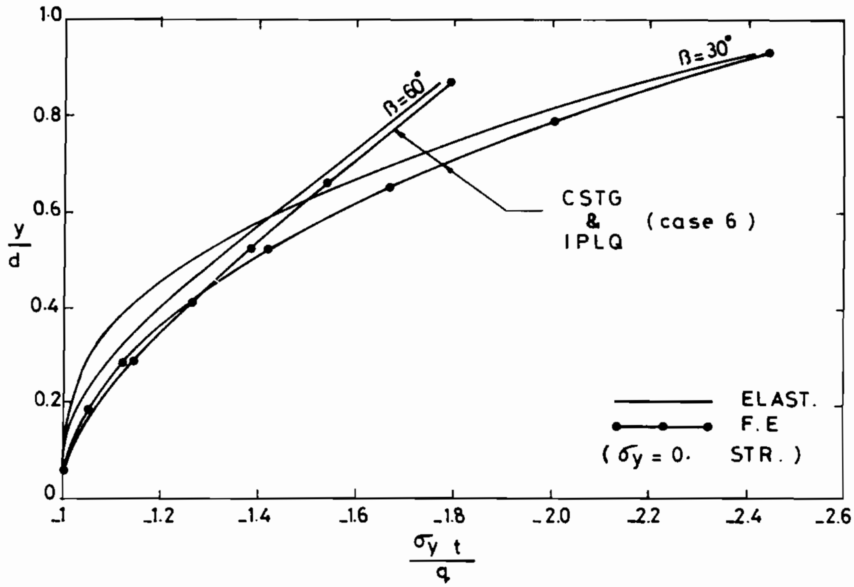


Fig. 16. Variation of σ_y for tapered beam.

elements used. The variation of the normal stresses σ_x, σ_y and the shear stress τ_{xy}, using the three methods of analysis on tapered cantilever beams with wedge angles, β, of 30 and 60° are presented in Figs 15, 16, and 17 respectively. Fig. 15 shows that values of σ_x obtained by the three methods of analysis are in agreement for both cases of β equal to 30 and 60°. The variation of the normal stress σ_y along the cross

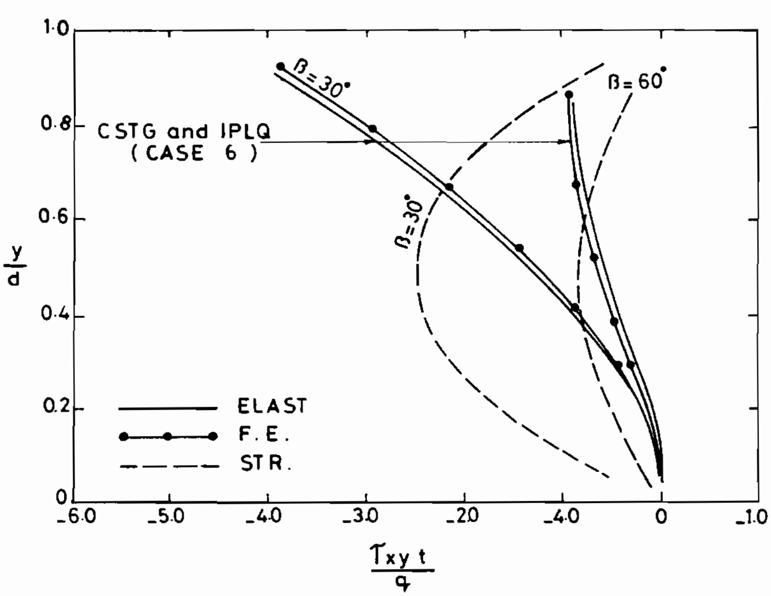


Fig. 17. Variation of τ_{xy} for tapered beam.

section of the beam shown in Fig. 16 indicates that the finite element solution yielded results about 10 percent higher than those obtained using the theory of elasticity. Fig. 17 shows that values of τ_{xy} as obtained by the strength of materials theory deviate significantly from results obtained from the theory of elasticity and the finite element approach.

CURVED BEAM SUBJECTED TO PURE BENDING

Consider a curved beam with inner radius a and outer radius b subjected to end moments M as shown in Fig. 18. At any radial distance r, the radial stress, σ_r, the circumferential stress, σ_θ, and the shear stresses, $\tau_{r\theta}$ can be calculated using the strength of materials theory as follows:

$$\sigma_r = \frac{M}{he(r_c - y)}\left[h \ln\left(\frac{r_c - y}{a}\right) - \left(\frac{h}{2} - y\right) \ln \frac{b}{a}\right] \quad (10a)$$

$$\sigma_\theta = \frac{M(y - e)}{Ae(r_c - y)} \quad (10b)$$

$$\tau_{r\theta} = 0 \quad (10c)$$

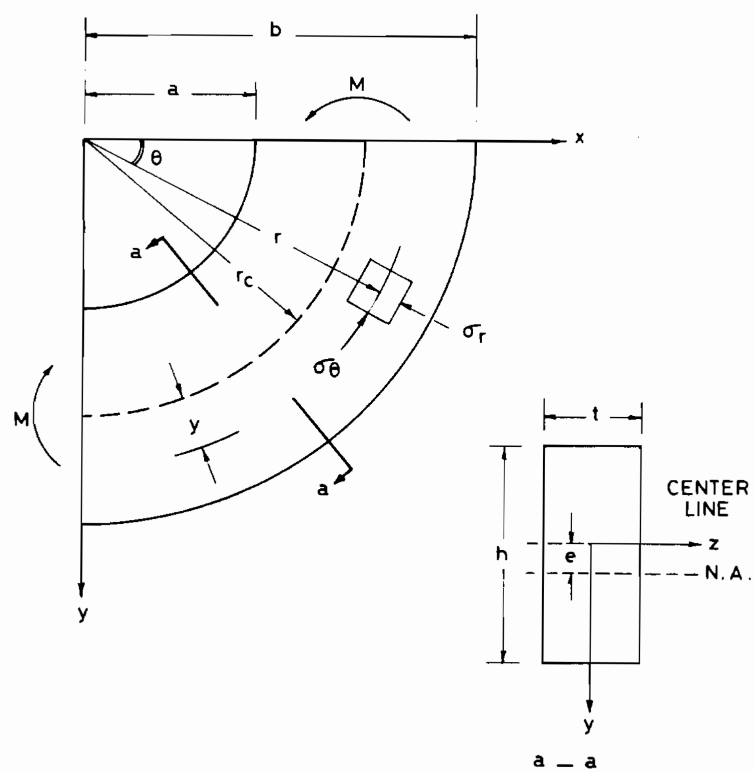


Fig. 18. Curved beam subjected to end moments.

where

$$e = r_c - \frac{h}{\ln\left(\frac{b}{a}\right)}$$

$$r_c = \frac{b+a}{2}$$

$$A = (b-a)t$$

h = depth of the beam

t = thickness

From the theory of elasticity (Popov 1978) the radial stress, σ_r, circumferential stress, σ_θ, and shear stresses, $\tau_{r\theta}$, can be expressed as

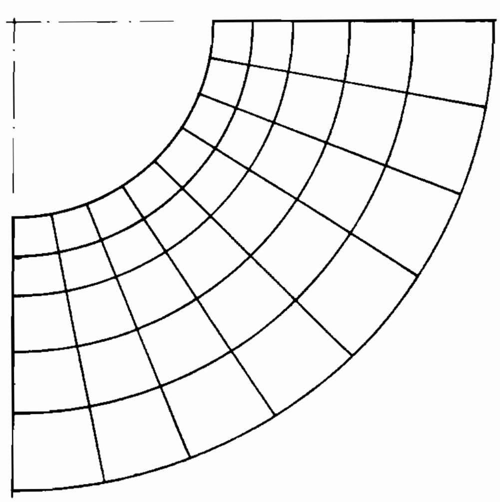
$$\sigma_r = -\frac{4M}{Dt}\left[\left(\frac{a^2b^2}{r^2}\ln\left(\frac{b}{a}\right) + b^2\ln\left(\frac{r}{b}\right) + a^2\ln\left(\frac{a}{r}\right)\right)\right] \tag{11a}$$

$$\sigma_\theta = \frac{4M}{Dt}\left[\frac{a^2b^2}{r^2}\ln\left(\frac{b}{a}\right) - b^2\ln\left(\frac{r}{b}\right) - a^2\ln\left(\frac{a}{r}\right) - (b^2 - a^2)\right] \tag{11b}$$

$$\tau_{r\theta} = 0 \tag{11c}$$

where $D = (b^2 - a^2)^2 - 4a^2b^2[\ln(b/a)]^2$, and t is the thickness of the beam which is assumed as unity in this analysis.

Three mesh sizes used in the STRUDL program for the analysis of this problem are shown in Fig. 19, and the different elements used in each mesh are given in Table 6.



Case 1. 209 NODES , 180 ELEMENTS.

Case 2. 361 NODES , 324 ELEMENTS.

Case 3. 532 NODES , 486 ELEMENTS.

Fig. 19. Typical element numbers and meshes for curved beam.

Table 6. Ratios of the radial stress σ_r and circumferential stress σ_θ for a curved beam subjected to end moments as computed by the finite element method to those computed by elasticity

Case	Type of element	No. of elements	No. of nodes	b/a	σ_r at $r/a = 2$ (F.E./Elast)	σ_θ at $r/a = 2$ (F.E./Elast.)
1	PSQ1	180	209	2·5	0·914	0·956
	PSR				0·927	0·966
	IPLQ				0·966	0·976
2	PSQ1	324	361	5	0·877	1·233
	PSR				0·819	1·164
	IPLQ				0·798	1·124
3	PSQ1	486	532	11	0·678	1·112
	PSR				0·633	1·050
	IPLQ				0·617	1·013

The variation of the radial stresses, σ_r, and the circumferential stresses, σ_θ, for curved beam subjected to pure bending using the three types of analysis are presented in Figs 20 and 21. These analyses are performed for beams with b/a ratio of 2·5, 5 and 11. Table 6 shows the ratios of both radial stresses and circumferential stresses as computed by the finite element method to that computed by the theory of elasticity at $r/a = 2$. The results indicate that with small radius of curvature (i.e. $b/a = 2{\cdot}5$) the ratios of the radial stresses and the circumferential stresses approach unity, indicating good agreement between the finite element and theory of elasticity approaches. Table 6 also shows that the IPLQ element in the STRUDL solution provides closer agreement with the theory of elasticity solution than PSR and PSQ1 elements. However for beams having larger radius of curvatures (i.e. $b/a = 5$ and 11) the accuracy of the radial stress σ_r as obtained by the STRUDL solution using the IPLQ, PSR and PSQ1 drops rapidly especially in case 3. Thus, increasing the number of elements and refinement of the mesh did not improve the accuracy of the radial stress σ_r. The circumferential stress σ_θ, however, seems to be fairly accurate for the cases analyzed.

The variation of the radial stresses σ_r calculated along the cross section of the beam using the strength of material theory, the theory of elasticity and the finite element solution is presented in Fig. 20. This figure shows that the variation of the radial stresses appears to be similar, but the magnitude for σ_r using the three approaches differ considerably. Results of the analysis shown in Fig. 21 indicate that the magnitude of the circumferential stress, σ_θ, as predicted by the three theories is almost the same.

CURVED CANTILEVER BEAM SUBJECTED TO CONCENTRATED FORCE

The last problem (Fig. 22) is the case of a curved cantilever beam subjected to a concentrated force at the free end. The radial, circumferential and shear stresses at any radial distance r from the origin, calculated by the strength of materials theory can be expressed as follows:

$$\sigma_r = 0 \tag{12a}$$

$$\sigma_\theta = \frac{P \sin\theta}{A} + \frac{Pr_c(y-e)\sin\theta}{Ae(r_c - y)} \tag{12b}$$

$$\tau_{r\theta} = 0 \tag{12c}$$

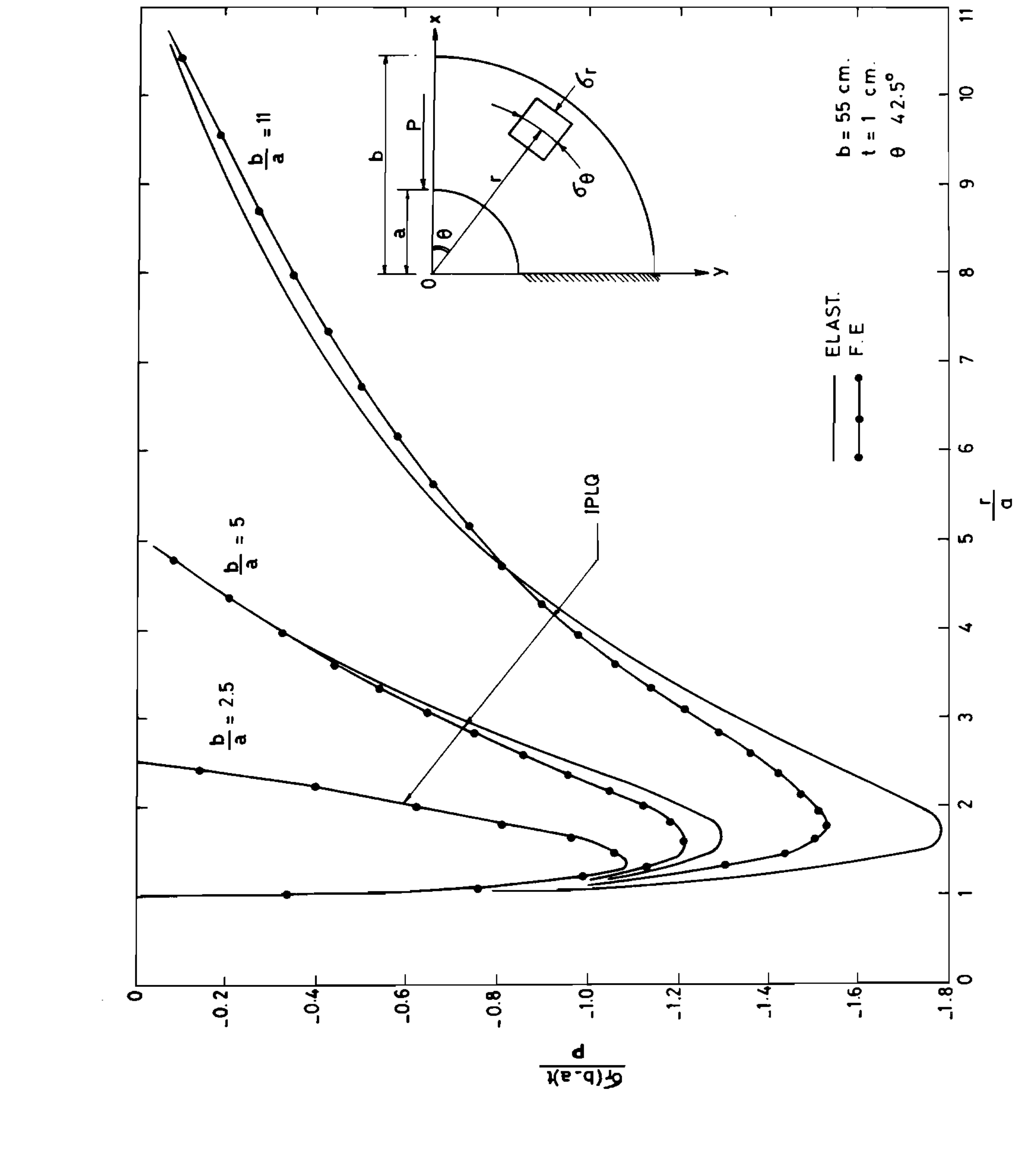


Fig. 20. Variation of σ_r for curved beam subjected to moment.

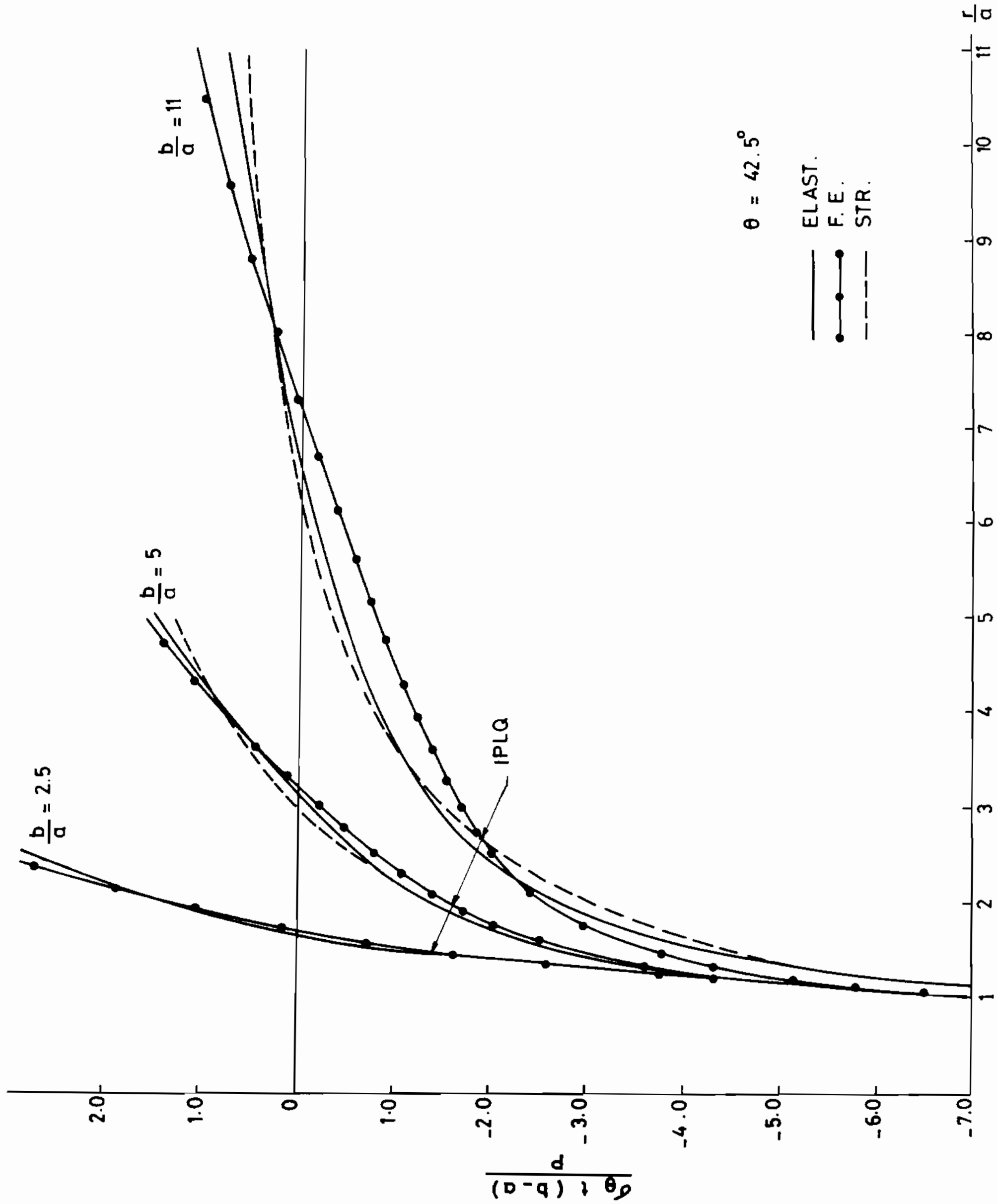


Fig. 21. Variation of σ_θ for curved beam subjected to moment.

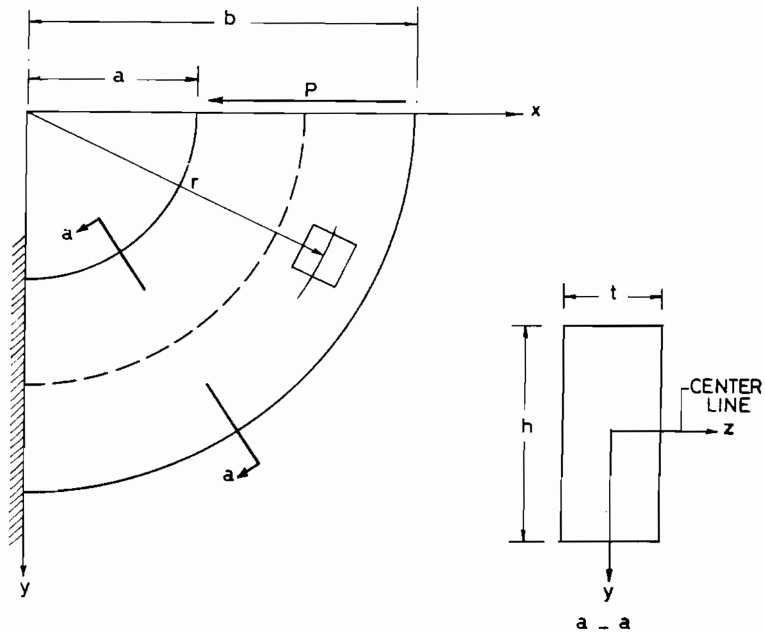


Fig. 22. Curved cantilever beam subjected to end force.

From the theory of elasticity the radial, circumferential and shear stresses can be expressed as

$$\sigma_r = \frac{P}{Nt}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\sin\theta \tag{13a}$$

$$\sigma_\theta = \frac{P}{Nt}\left(3r - \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\sin\theta \tag{13b}$$

$$\tau_{r\theta} = -\frac{P}{Nt}\left(r + \frac{a^2b^2}{r^3} - \frac{a^2+b^2}{r}\right)\cos\theta \tag{13c}$$

where $N = a^2 - b^2 + (a^2 + b^2)\ln(b/a)$, and t is the thickness of the beam which is assumed to be unity.

As in the previous case, the three types of mesh used in the STRUDL program for the analysis of this problem are shown in Fig. 19. Table 7 shows the different types of element used in the analyses for curved beams with b/a ratio of 2·5, 5 and 11.

The ratios of the radial and circumferential stresses, for $r/a = 2$, computed by the finite element method to that computed by the theory of elasticity are presented in Table 7. For curved beam analyzed with $b/a = 2{\cdot}5$ the results show that the ratios of radial and circumferential stresses as obtained by the finite element analysis and the theory of elasticity approach unity. This indicates that there is good agreement between results of the finite element and those of the theory of elasticity. Results of the analyses also indicate that the IPLQ element in case 1 performs, in the STRUDL solution, better than elements PSR and PSQ1. However, increasing the

Table 7. Ratios of the radial stress σ_r and circumferential stress σ_θ for a curved beam subjected to end moments as computed by the finite element method to those computed by elasticity

Case	Type of element	No. of elements	No. of nodes	b/a	σ_r at $r/a = 2$ (F.E./Elast.)	σ_θ at $r/a = 2$ (F.E./Elast.)
1	PSQ1	180	209	2·5	0·914	0·956
	PSR				0·927	0·966
	IPLQ				0·966	0·976
2	PSQ1	324	361	5	0·877	1·233
	PSR				0·819	1·164
	IPLQ				0·798	1·124
3	PSQ1	486	532	11	0·678	1·112
	PSR				0·633	1·050
	IPLQ				0·617	1·013

radius of curvature of the curved beam, i.e. $b/a = 5$ and 11, causes σ_r, obtained by the finite element analysis to drop rapidly regardless of the number of elements used. Thus, increasing the number of elements and refinement of the mesh did not improve the accuracy of the radial stress σ_r. However, results of the analysis indicate that the circumferential stress σ_θ is fairly accurate and almost insensitive to mesh size.

Variations of radial stresses σ_r for different types of curved beam are shown in Fig. 23. This figure indicates that for curved beams with small radius of curvature (i.e. $b/a = 2{\cdot}5$) the theory of elasticity and the finite element solutions give similar results. However, when the ratio of b/a increases (i.e. $b/a = 5$ and 11) the pattern of the radial stress σ_r as predicted by the two solutions remains the same, but the magnitude of stresses obtained differ considerably. Fig. 24 shows the variation of the tangential stresses σ_θ for different types of curved beam. Results presented in this figure indicate good agreement between the values of σ_θ obtained by the three methods of analysis for curved beams with b/a equal to 2·5. However, for beams with b/a of 5 and 11 the results diverge a little. Fig. 25 shows the variation of the shear stress $\tau_{r\theta}$ for different types of curved beam. Again the results indicate good agreement between the theory of elasticity and the finite element solution for curved beams with b/a ratio equal to 2·5, while for beams with b/a equal 5 and 11, variation in pattern of $\tau_{r\theta}$ remains the same, but the numerical values are higher when the finite element analysis is used.

DISCUSSION OF RESULTS

Six examples are presented in this study to determine the differences in magnitude and distribution of stresses as predicted by three approaches, the strength of materials theory, the theory of elasticity, and the finite element method. These examples clearly show that the approximate strength of materials theory cannot be used to determine stresses normal to the axis of the beam (i.e. σ_y) in both straight and tapered beams. The strength of materials theory can provide accurate information regarding bending stress (i.e. σ_x) along the axis of straight beams as well as tapered beams that can be used in design. However, the strength of materials theory can provide accurate values of τ_{xy} for straight beams with rectangular cross sections but only approximate values for beams with nonuniform cross section. For curved

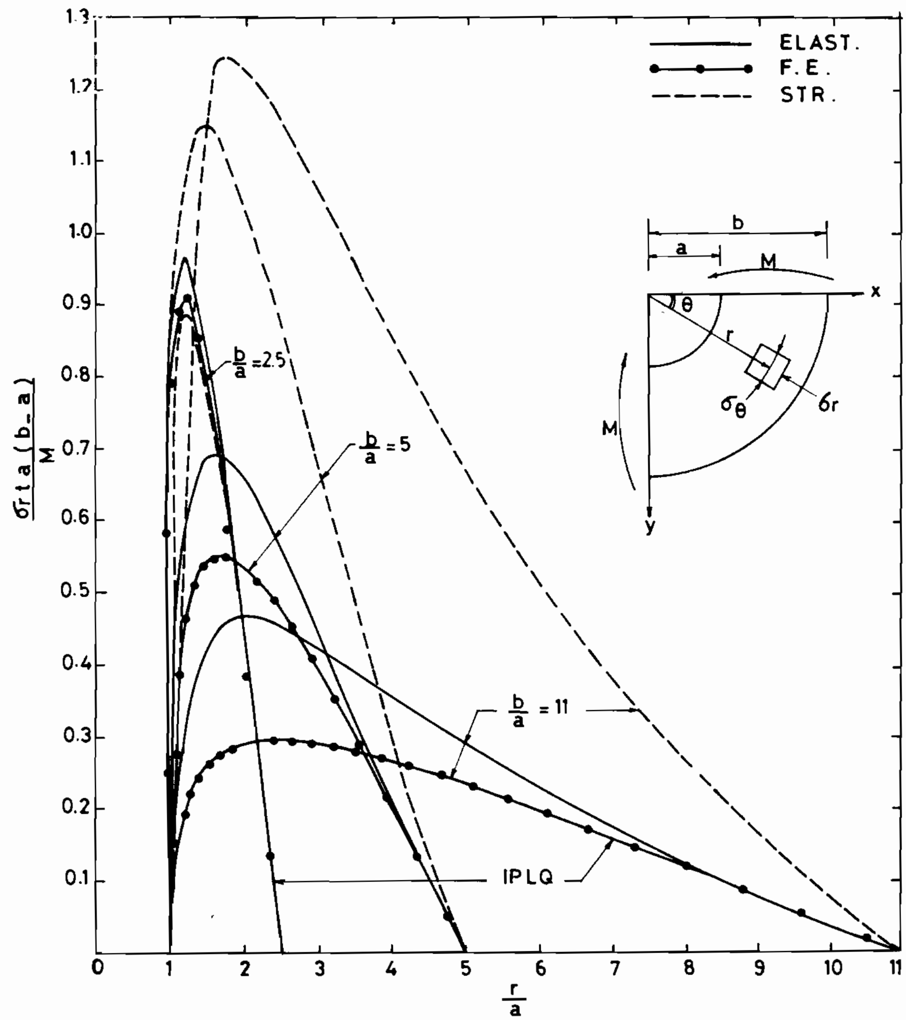


Fig. 23. Variation of σ_r for curved beam subjected to end force.

beams the strength of materials theory provides only approximate values for stresses in comparison to those obtained by the theory of elasticity.

Results obtained from the finite element analysis clearly demonstrate that the magnitude of stresses are significantly influenced by the types of element used, geometry of the mesh, and number of elements used in the solution. From Tables 2, 3, and 4 it is shown that PSR elements with aspect ratio of one, provide satisfactory results for straight beams having rectangular cross sections. These tables show that increasing the number of elements in the mesh beyond a certain limit will not improve the accuracy of the analysis. Thus any increase of the number of elements beyond that limit is not justified. By examining Table 5 case 1, we can see that PSR elements did not provide satisfactory results for tapered beams. The standard quadrilateral-isoparametric element IPLQ provides satisfactory results for straight

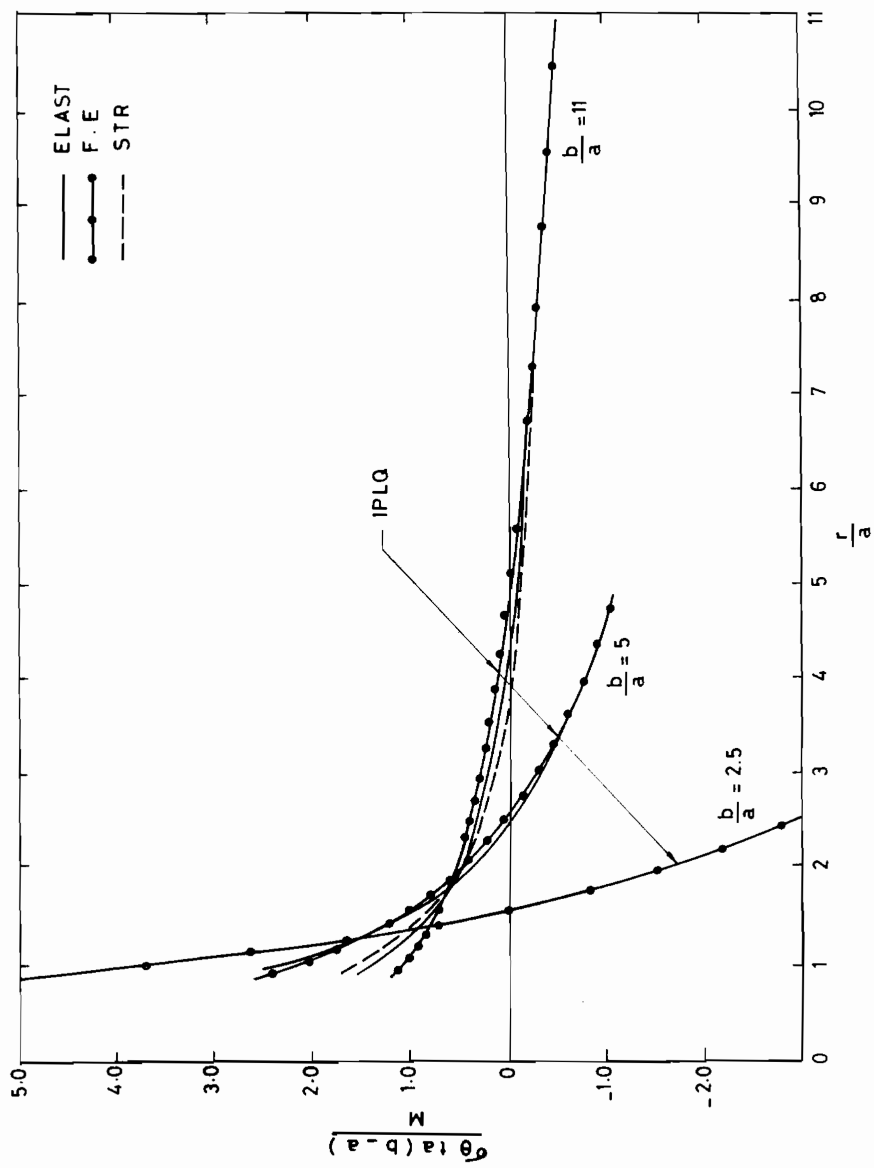


Fig. 24. Variation of σ_θ for curved beam subjected to end force.

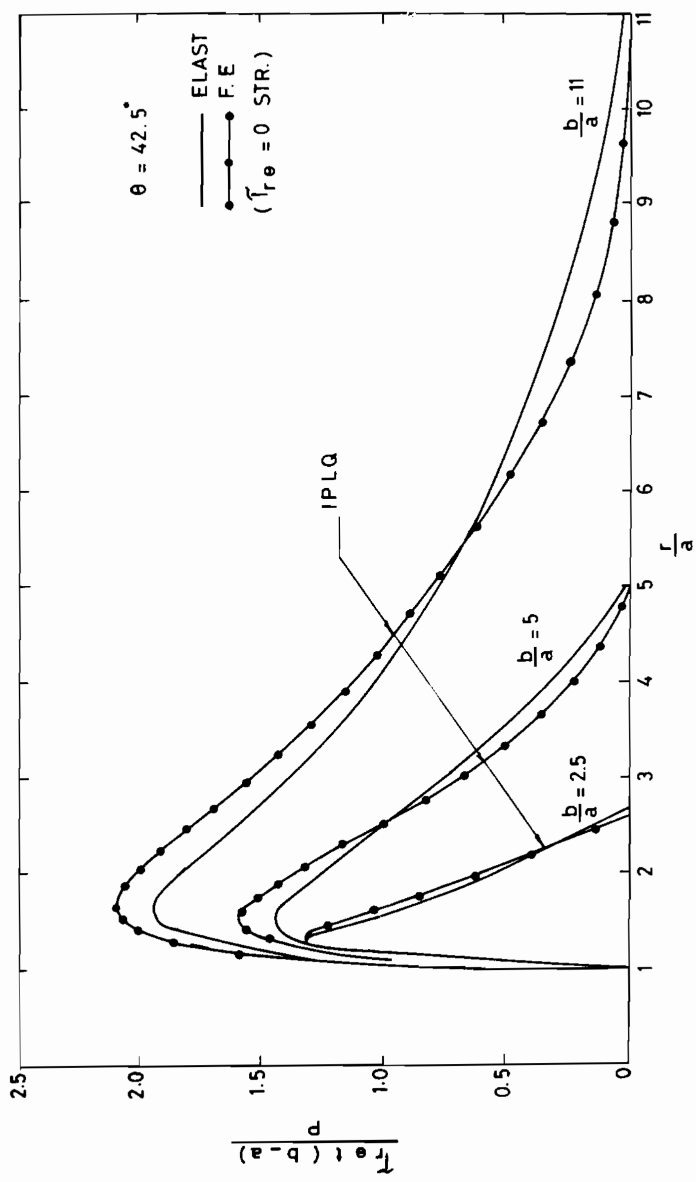


Fig. 25. Variation of $\tau_{r\theta}$ for curved beam subjected to end force.

beam, cases 1 and 2, and tapered beam case 3. Trying to improve the results by doubling the number of elements as shown in case 6 did not provide significantly better results. Using the higher order quadrilateral-isoparametric element IPQQ rendered similar results as those of the IPLQ, but required more mesh refinement as indicated in cases 7 and 8. The PSQ1 did not provide accurate results which were also found to be insensitive to mesh refinement as indicated in cases 2 and 5. Figs 23, 24, and 25 show that although element IPLQ renders good results for cases of curved beam subjected to end force having b/a ratio equal 2·5, it is not recommended for curved beams having higher values of b/a, i.e. $b/a = 5$ or 11.

CONCLUSIONS

In the previous six examples it is clearly demonstrated that the simple strength of materials theory provides useful and reliable information for many engineering problems. However, the applicability and usefulness of the theory is dictated by the geometry of the problem. The strength of materials theory provides bending stresses comparable to those predicted by the theory of elasticity. However, the shear stresses deviate significantly from the theory of elasticity for nonprismatic beams or curved beams.

The accuracy of the finite element results is influenced by the type of element used, geometry of the mesh, and the number of elements used. The analysis showed that the increase of the number of elements beyond a certain limit did not improve the accuracy of the results. Of all the elements selected for the finite element analysis (i.e. triangular, square, quadrilateral, and isoparametric elements with different order of complexity) there is no single element that can provide accurate results for every problem.

It is hoped that this study will make engineers aware of the advantages and limitations of stress analysis used in design and also provide the computer programmer with better understanding of the element that the analyst can select for design purposes.

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مقارنة بين نتائج نظرية مقاومة المواد ونظريتي المرونة والعناصر الجزئية

غزوان نايف و مساعد الحسيني
قسم الهندسة المدنية بجامعة الكويت ،
ص . ب . ٥٩٦٩ ، الصفاة ١٣٠٦٠ ، الكويت

خلاصة

ان مقدار الاجهادات وطريقة توزيعها يمكن الحصول عليها باستخدام نظرية مقاومة المواد او نظرية المرونة او نظرية العناصر الجزئية . وهذا البحث يشتمل على مقارنة النتـائج التي أمكن الحصول عليها بهذه النظريات الثلاث ، باستخدام عدد من النماذج والمسائل الهندسية البسيطة . وقد وضعت النتائج على شكل بيانات وجداول لامكان مقارنتها مع بعضها بسهولة واستخلاص التوصيات منها .

اوضحت النتائج أن القيمة العددية للاجهادات تعتمد على النظرية المستعملة ، وأن الفروق في النتائج تعكس الاختلاف في الفرضيات التي بنيت عليها كل نظرية . وعندما تظل المقاطع بعد الأحمال مستوية مثلما كانت قبل الاحمال ، فقد وجد أن نظرية مقاومة المواد البسيطة تعطي نتائج جيدة وقريبة جدا من نتائج النظريات الأخرى الأكثر تعقيدا .